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Semi-direct Galois covers of the affine line

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Abstract. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $G$ be a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ where $b$ is a positive integer and $\ell$ is a prime distinct from $p$. In this paper, we study Galois covers $\psi: Z \rightarrow \mathbb{P}_k^1$ ramified only over $\infty$ with Galois group $G$. We find the minimal genus of a curve $Z$ which admits a covering map of this form and we give an explicit formula for this genus in terms of $\ell$ and $p$. The minimal genus occurs when $b$ equals the order $a$ of $\ell$ modulo $b$ and we also prove that the number of curves $Z$ of this minimal genus which admit such a covering map is at most $(p-1)/a$ when $p$ is odd.

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1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. In sharp contrast with the situation in characteristic 0, there exist Galois covers $\psi : Z \to \mathbb{P}_k^1$ ramified only over infinity. By Abhyankar’s Conjecture [2], proved by Raynaud and Harbater [9], [4], a finite group $G$ occurs as the Galois group of such a cover $\psi$ if and only if $G$ is quasi-$p$, i.e., $G$ is generated by $p$-groups. This result classifies all the finite quotients of the fundamental group $\pi_1(\mathbb{A}_k^1)$. It does not, however, determine the profinite group structure of $\pi_1(\mathbb{A}_k^1)$ because this fundamental group is an infinitely generated profinite group.

There are many open questions about Galois covers $\psi : Z \to \mathbb{P}_k^1$ ramified only over infinity. For example, given a finite quasi-$p$ group $G$, what is the smallest integer $g$ for which there exists a cover $\psi : Z \to \mathbb{P}_k^1$ ramified only over infinity with $Z$ of genus $g$? As another example, suppose $G$ and $H$ are two finite quasi-$p$ groups such that $H$ is a quotient of $G$. Given an unramified Galois cover $\phi$ of $\mathbb{A}_k^1$ with group $H$, under what situations can one dominate $\phi$ with an unramified Galois cover $\psi$ of $\mathbb{A}_k^1$ with Galois group $G$? Answering these questions will give progress towards understanding how the finite quotients of $\pi_1(\mathbb{A}_k^1)$ fit together in an inverse system. These questions are more tractible for quasi-$p$ groups that are $p$-groups since the maximal pro-$p$ quotient $\pi^p(\mathbb{A}_k^1)$ is free (of infinite rank) [10].

In this paper, we study Galois covers $\psi : Z \to \mathbb{P}_k^1$ ramified only over $\infty$ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell \mathbb{Z})^b \times \mathbb{Z}/p\mathbb{Z}$, where $\ell$ is a prime distinct from $p$. Such a cover $\psi$ must be a composition $\psi = \phi \circ \omega$ where $\omega : Z \to Y$ is unramified and $\phi : Y \to \mathbb{P}_k^1$ is an Artin-Schreier cover ramified only over $\infty$. The cover $\phi$ has an affine equation $y^p - y = f(x)$ for some $f(x) \in k[x]$ with degree $s$ prime-to-$p$. The $\ell$-torsion $\text{Jac}(Y)[\ell]$ of the Jacobian of $Y$ is isomorphic to $(\mathbb{Z}/\ell \mathbb{Z})^{2g_Y}$. When $f(x) = x^s$, we determine how an automorphism $\tau$ of $Y$ of order $p$ acts on $\text{Jac}(Y)[\ell]$. This allows us to construct a Galois cover $\psi_a : Z_a \to \mathbb{P}_k^1$ ramified only over $\infty$ which dominates $\phi$, such that the Galois group of $\psi_a$ is $(\mathbb{Z}/\ell \mathbb{Z})^b \times \mathbb{Z}/p\mathbb{Z}$ where $a$ is the order of $\ell$ modulo $p$ (Section 3). We prove that the genus of $Z_a$ is minimal among all natural numbers that occur as the genus of a curve $Z$ which admits a covering map $\psi : Z \to \mathbb{P}_k^1$ ramified only over $\infty$ with Galois group of the form $(\mathbb{Z}/\ell \mathbb{Z})^b \times \mathbb{Z}/p\mathbb{Z}$. We also prove that the number of curves $Z$ of this minimal genus which admit such a covering map is at most $(p - 1)/a$ when $p$ is odd (Section 4).

2 Quasi-$p$ semi-direct products

We recall which groups occur as Galois groups of covers of $\mathbb{P}_k^1$ ramified only over $\infty$.

**Definition 2.1** A finite group is a quasi-$p$-group if it is generated by all of its Sylow $p$-subgroups.

It is well-known that there are other equivalent formulations of the quasi-$p$ property, such as the next result.

**Lemma 2.2** A finite group is a quasi-$p$-group if and only if it has no nontrivial quotient group whose order is relatively prime to $p$.

The importance of the quasi-$p$ property is that it characterizes which finite groups occur as Galois groups of unramified covers of the affine line.

**Theorem 2.3** A finite group occurs as the Galois group of a Galois cover of the projective line $\mathbb{P}_k^1$ ramified only over infinity if and only if it is a quasi-$p$ group.
Suppose a quasi-

If

Part (1) is true since

Moreover, 

Then

This is a special case of Abhyankar’s Conjecture [2] which was jointly proved by Harbater [4] and Raynaud [9].

We now restrict our attention to groups $G$ that are semi-direct products of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. The semi-direct product action is determined by a homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^b)$.

**Lemma 2.4** Suppose a quasi-$p$ group $G$ is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ for a positive integer $b$.

1. Then $G$ is not a direct product.
2. Moreover, $b \geq \text{ord}_p(\ell)$ where $\text{ord}_p(\ell)$ is the order of $\ell$ modulo $p$.

**Proof** Part (1) is true since $(\mathbb{Z}/\ell\mathbb{Z})^b$ cannot be a quotient of the quasi-$p$ group $G$. For part (2), the structure of a semi-direct product $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ depends on a homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^b)$. By part (1), $\iota$ is an inclusion. Thus $\text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^b) \simeq \text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})$ has an element of order $p$. Now

$$|\text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})| = (\ell^b - 1)(\ell^b - \ell) \cdots (\ell^b - \ell^{b-1}).$$

Thus $\ell^b \equiv 1 \mod p$ for some positive integer $\beta \leq b$ which implies $b \geq \text{ord}_p(\ell)$.

**Lemma 2.5** If $a = \text{ord}_p(\ell)$, then there exists a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ which is quasi-$p$. It is unique up to isomorphism.

**Proof** If $a = \text{ord}_p(\ell)$, then there is an element of order $p$ in $\text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^a)$ and so there is an injective homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^a)$. Thus there exists a non-abelian semi-direct product $G$ of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. To show that $G$ is quasi-$p$, suppose $N$ is a normal subgroup of $G$ whose index is relatively prime to $p$. Then $N$ contains an element $\tau$ of order $p$. By [3, 5.4, Thm. 9], since $G$ is not a direct product and $(\mathbb{Z}/\ell\mathbb{Z})^a$ is normal in $G$, the subgroup $\langle \tau \rangle$ is not normal in $G$. Thus $\langle \tau \rangle$ is a proper subgroup of $N$. It follows that $\ell$ divides $|N|$ and so $N$ contains an element $h$ of order $\ell$ by Cauchy’s theorem. Recall that $\text{Aut} ((\mathbb{Z}/\ell\mathbb{Z})^\beta)$ contains no element of order $p$ for any positive integer $\beta < a$. Thus the group generated by the conjugates of $h$ under $\tau$ has order divisible by $\ell^a$. Thus $N = G$ and $G$ has no non-trivial quotient group whose order is relatively prime to $p$. By Lemma 2.2, $G$ is quasi-$p$.

The uniqueness follows from [8, Lemma 6.6].

**3 Explicit construction of $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$-Galois covers of $\mathbb{A}^1_k$**

In this section, we give concrete examples of Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over $\infty$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. To compute the genus of the covering curve $Z$, we will need to determine the higher ramification groups of $\psi$.

**Definition 3.1** Let $L/K$ be a Galois extension of function fields of curves with Galois group $G$ and let $P, P'$ be primes of $K$ and $L$ such that $P'|P$. Let $\nu_{P'}$ and $\mathcal{O}_{P'}$ be the corresponding valuation function and valuation ring for $P'$. For any integer $i \geq -1$, the $i$th ramification group of $P'|P$ is

$$I_i(P'|P) = \{ \sigma \in G \mid \nu_{P'}(\sigma(z) - z) \geq i + 1, \forall z \in \mathcal{O}_{P'} \}.$$ 

**Lemma 3.2** Suppose $f(x) \in k[x]$ is a polynomial of degree $s$ for a positive integer $s$ prime to $p$. Let $\phi : Y \to \mathbb{P}^1_k$ be the cover of curves corresponding to the field extension

$$k(x) \leftarrow k(x)[y]/(y^p - y - f(x)).$$
1. Then \( \phi : Y \rightarrow \mathbb{P}^1_k \) is a Galois cover with Galois group \( \mathbb{Z}/p\mathbb{Z} \) ramified only at the point \( P_\infty \) over \( \infty \).

2. The \( i \)th ramification group at \( P_\infty \) satisfies
\[
I_i = \begin{cases} 
\mathbb{Z}/p\mathbb{Z} & \text{if } i \leq s \\
0 & \text{if } i > s.
\end{cases}
\]

3. The genus \( g_Y \) of \( Y \) is equal to
\[
g_Y = (p - 1)(s - 1)/2.
\]

**Proof** For part (1), note that the extension \( k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)) \) is cyclic of degree \( p \), with Galois group generated by the automorphism \( \tau : y \mapsto y + 1 \) of order \( p \). Let \( P \) be a finite prime of \( k(x) \) and let \( \nu_P \) be the corresponding valuation. Then \( \nu_P(f(x)) \geq 0 \), hence \( P \) is unramified by [12, Prop. III.7.8(b)]. For the infinite prime \( \infty \) with corresponding valuation \( \nu_\infty \), we have
\[
\nu_\infty(f(x) - (z^p - z)) \leq 0
\]
for all \( z \in k[x] \) thus \( P_\infty \) is totally ramified by [12, Prop. III.7.8(c)].

To prove part (2), we note that furthermore
\[
v_{P_\infty}(y^p - y) = v_{P_\infty}(f(x)) = v_{P_\infty}(x^s) = -sp,
\]
which implies that
\[
v_{P_\infty}(y) = -s.
\]

Now let \( \hat{\theta} \) be the completion of the valuation ring of \( k(x)[y]/(y^p - y - f(x)) \) at \( P_\infty \), and let \( \pi_\infty \) be a generator of the unique prime in \( \hat{\theta} \). Then write \( y = \pi_\infty^{-s}u \), where \( u \) is a unit in \( \hat{\theta} \cong k[[\pi_\infty]] \). Since \( k \) is algebraically closed, \( \sqrt{u} \in \hat{\theta} \), and so \( \sqrt{y} \in \hat{\theta} \). After possibly changing \( \pi_\infty \), we can assume without loss of generality that \( \sqrt{y} = \pi_\infty^{-1} \). Recalling that \( \tau \) acts on \( y \) by \( \tau(y) = y + 1 \), we have
\[
\tau(\pi_\infty) = \frac{\tau(1)}{\pi_\infty^{1/s}} = \left(\frac{\pi_\infty^{1/s}/(1 + \pi_\infty^{1/s})}{\pi_\infty^{1/s}}\right)^{1/s}
\]
\[
= \pi_\infty(1 - \pi_\infty^{1/s} + \pi_\infty^{2s} - + \ldots)^{1/2}
\]
\[
= \pi_\infty - (1/s)\pi_\infty^{s+1} + a_{2s+1}\pi_\infty^{2s+1} - + \ldots.
\]
Thus \( v_{P_\infty}(\tau(\pi_\infty) - \pi_\infty) = s + 1 \), which completes the proof of part (2).

To find the genus \( g_Y \) of \( Y \) for part (3), we make use of the Riemann-Hurwitz formula
\[
2g_Y - 2 = p(-2) + \sum_{i=0}^{\infty} (|I_i| - 1),
\]
where \( I_i \) denotes the \( i \)th ramification group at \( P_\infty \), [5, Thms. 7.27 & 11.72]). From part (2), we then obtain that \( g_Y = (p - 1)(s - 1)/2 \).

Recall the following facts about the \( p \)th cyclotomic polynomial \( \Phi_p(t) := t^{p-1} + \ldots + 1 \), which is the minimal polynomial over \( \mathbb{Q} \) of a primitive \( p \)th root of unity \( \zeta_p \). Now \( \mathbb{Q}(\zeta_p) \) is a Galois extension of \( \mathbb{Q} \), unramified over \( \ell \) since \( \ell \neq p \), and all primes over \( \ell \) have the same residue field degree. The irreducible factors of \( \Phi_p(t) \) modulo \( \ell \) are in one-to-one correspondence with the primes of \( \mathbb{Z}/[\zeta_p] \) over \( \ell \), and each of their degrees is equal to the residue field degree of the corresponding prime over \( \ell \). The latter equals the order \( a = \text{ord}_p(\ell) \) of \( \ell \) modulo \( p \) [3, Ch. 12.2, Exercise #20].

We shall soon explicitly construct a cover of \( \mathbb{P}^1_k \) ramified only over \( \infty \) with Galois group \( (\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z} \). But before we do so, we start with a specific example.
Example 3.3 Let $p$ be an odd prime. Consider the Artin-Schreier cover $\phi : Y_2 \rightarrow \mathbb{P}^1_k$ corresponding to the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s)$. By Lemma 3.2(3), the genus of $Y_2$ is $g_Y = (p - 1)/2$.

Let $\text{Jac}(Y)$ be the Jacobian of $Y$. The automorphism $\tau$ of $Y$ given by $\tau(y) = y + 1$ defines an automorphism of $\text{Jac}(Y)$ of order $p$.

Now we describe the action of $\tau$ on the subgroup $\text{Jac}(Y)[2]$ of 2-torsion points of $\text{Jac}(Y)$ explicitly. Note that since $2g_Y = (p - 1)$, then $\text{Jac}(Y)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ by [7, pg. 64]. Thus we can represent $\tau$ as an element of $\text{GL}_{p-1}(\mathbb{Z}/2\mathbb{Z})$.

For $0 \leq i \leq p - 1$, let $P_i$ denote the closed point of $Y$ at which the function $y - i$ vanishes. For each $i$, the divisors $P_i$ and $D_i = P_i - P_\infty$ on $Y$ can be identified with elements of $\text{Jac}(Y)$. Let $O$ be the identity element of $\text{Jac}(Y)$, i.e., the linear equivalence class of principal divisors. Then the divisor $2D_i$ is equivalent to $O$ since $\text{div}(y - i) = 2D_i$. Moreover since $\text{div}(x) = D_0 + D_1 + \cdots + D_{p-1}$ is equivalent to 0, we have $D_i \in \text{Jac}(Y)[2]$ with the only relation $D_{p-1} = -(D_0 + D_1 + \cdots + D_{p-2})$. In particular, $D_0, \ldots, D_{p-2}$ form a basis of $\text{Jac}(Y)[2]$. With respect to this basis, the action of $\tau$ can be represented by the $(p - 1) \times (p - 1)$-matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & -1 \\
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1
\end{pmatrix}.
$$

The characteristic polynomial of $\tau$ is $\Phi_\tau(t) = 1 + t + \cdots + t^{p-1} \in (\mathbb{Z}/2\mathbb{Z})[t]$, which factors into irreducible polynomials each of degree equaling the order of 2 modulo $p$. In particular, $\tau$ acts irreducibly on $\text{Jac}(Y)[2]$ if and only if 2 is a primitive root modulo $p$, i.e., if and only if $p$ is an Artin prime.

For example, if $p = 3$, then $\tau$ acts irreducibly on $\text{Jac}(Y)[2]$ with minimal polynomial $\Phi_\tau(t) = t^2 + t + 1$. If $p = 7$, then 2 has order 3 modulo 7 and the factorization of $\Phi_\tau(t)$ into irreducible polynomials is $\Phi_\tau(t) \equiv (x^3 + x^2 + 1)(x^3 + x + 1)$ modulo 2. Thus the action of $\tau$ on $\text{Jac}(Y)[2]$ can be represented by the $6 \times 6$-matrix

$$
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
$$

where $A_1$ and $A_2$ are the irreducible 3-dimensional companion matrices of $x^3 + x^2 + 1$ and $x^3 + x + 1$ respectively.

For the rest of the paper, let $\phi_s : Y_s \rightarrow \mathbb{P}^1_k$ be the Artin-Schreier cover corresponding to the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s)$. We show that $\phi_s$ can be dominated by a Galois cover of $\mathbb{P}^1_k$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ for $a$ equal to the order of $\ell$ modulo $p$.

Proposition 3.4 Let $s$ and $\ell$ be primes distinct from $p$. Let $\phi_s : Y_s \rightarrow \mathbb{P}^1_k$ be the Artin-Schreier cover with affine equation $y^p - y = x^s$. Let $a = \text{ord}_p(\ell)$ be the order of $\ell$ modulo $p$. Then there exists an unramified Galois cover $\omega : \mathbb{Z}_a \rightarrow Y_s$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a$ such that $\psi_a = \phi_s \circ \omega : \mathbb{Z}_a \rightarrow \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ ramified only over $\infty$ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$.

Proof By Lemma 3.2(1), $\phi_s : Y_s \rightarrow \mathbb{P}^1_k$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point $P_\infty$ over $\infty$. The genus $g_s$ of $Y_s$ is $(p - 1)(s - 1)/2$. Consider
two commuting automorphisms of \(Y_s\) defined by
\[
\tau : \begin{cases} 
  x \mapsto x, \\
  y \mapsto y + 1,
\end{cases} \quad \sigma : \begin{cases} 
  x \mapsto \zeta_s x, \text{ where } \zeta_s \text{ is a primitive } s\text{th root of unity}, \\
  y \mapsto y.
\end{cases}
\]

Let \(\text{Jac}(Y_s)\) be the Jacobian of \(Y_s\). Then \(\tau\) and \(\sigma\) define commuting automorphisms of \(\text{Jac}(Y_s)\) of orders \(p\) and \(s\) respectively. Therefore, \(\text{End}(\text{Jac}(Y_s))\) contains a ring isomorphic to \(\mathbb{Z}[\zeta_p, \zeta_s] \cong \mathbb{Z}[\zeta_{ps}]\), which is a \(\mathbb{Z}\)-module of rank \(\phi(ps) = (p-1)(s-1) = 2g_s\). Then \(\mathbb{Q}(\zeta_{ps})\) is contained in \(\text{End}(\text{Jac}(Y_s)) \otimes \mathbb{Q}\). In other words, \(\text{Jac}(Y_s)\) has complex multiplication by \(\mathbb{Q}(\zeta_{ps})\).

For a prime \(\ell\) distinct from \(p\), the automorphism \(\tau\) induces an action on the subgroup \(\text{Jac}(Y_s)[\ell]\) of \(\ell\)-torsion points of \(\text{Jac}(Y_s)\). Recall that there is a bijection between \(\ell\)-torsion points \(D\) of \(\text{Jac}(Y_s)\) and unramified \((\mathbb{Z}/\ell\mathbb{Z})\)-Galois covers \(\omega_D : Z_D \to Y_s\) [6, Prop. 4.11]. Also \(D\) has order \(\ell\) if and only if \(Z_D\) is connected. This induces a bijection between orbits of \(\tau\) on the set of unramified \((\mathbb{Z}/\ell\mathbb{Z})\)-Galois covers \(\omega_D : Z_D \to Y_s\) and on the set of \(\ell\)-torsion points of \(\text{Jac}(Y_s)\). For a point \(D\) of order \(\ell\) of \(\text{Jac}(Y_s)\), consider the compositum \(\omega : Z \to Y_s\) of all of the conjugates \(\omega_{\tau^j(D)} : Z_{\tau^j(D)} \to Y_s\) for \(0 \leq j \leq p - 1\):

\[
\begin{array}{c}
\text{Z}_D \quad \text{Z}_{\tau(D)} \quad \cdots \quad \text{Z}_{\tau^{p-1}(D)} \\
\downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
\text{Y}_s \quad \text{Y}_s \quad \cdots \quad \text{Y}_s
\end{array}
\]

Then \(Z\) is invariant under \(\tau\) and so \(\phi_s \circ \omega : Z \to \mathbb{P}^1_k\) is Galois. Moreover, \(\phi_s \circ \omega\) is the Galois closure of \(\phi_s \circ \omega_D : Z_D \to \mathbb{P}^1_k\).

Suppose there is a non-trivial one-dimensional \(\tau\)-invariant subspace of \(\text{Jac}(Y_s)[\ell]\) with eigenvalue \(1\); i.e. \(\tau\) acts trivially on this subgroup of order \(\ell\). This yields a cover \(\psi_s \circ \omega_1 : Z_1 \to Y_s \to \mathbb{P}^1_k\). Since the action of \(\tau\) is trivial, \(\psi_s \circ \omega_1\) is Galois, ramified only over \(\mathbb{Q}\), with abelian Galois group \(\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\). This contradicts Lemma 2.4.

Since \(\tau\) has order \(p\), the minimal polynomial \(m_\tau(t)\) of \(\tau\) divides \(t^p - 1 = (t - 1)(t^{p-1} + \cdots + 1)\) in \((\mathbb{Z}/\ell\mathbb{Z})[t]\). From the preceding paragraph, there is no non-trivial one-dimensional \(\tau\)-invariant subspace of \(\text{Jac}(Y_s)[\ell]\) with eigenvalue \(1\). This implies that \(m_\tau(t)\) divides the \(p\)th cyclotomic polynomial \(\Phi_p(t) = t^p - 1 + \cdots + 1\) in \((\mathbb{Z}/\ell\mathbb{Z})[t]\). The irreducible factors of \(\Phi_p(t)\) in \((\mathbb{Z}/\ell\mathbb{Z})[t]\) all have degree \(a\). Thus the degree of \(m_\tau(t)\) equals \(a\).

Since \(2g_s = (p-1)(s-1)\), we have \(\text{Jac}(Y_s)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{(p-1)(s-1)}\), so we can represent \(\tau\) as an element of \(\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})\). We can choose a basis of \(\text{Jac}(Y_s)[\ell]\) such that \(\tau\) is represented as an element of \(\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})\) in block form. The first irreducible subrepresentation of \(\tau\) has dimension \(a\). Moreover, since \(\mathbb{Q}(\zeta_{ps})\) is a Galois extension of \(\mathbb{Q}\), the block form of \(\tau\) consists entirely of irreducible blocks of the same size. In particular, the number of irreducible blocks is \((p-1)(s-1)/a\). In other words, \(\tau\) can be represented by an element of \(\text{GL}_{(s-1)(p-1)}(\mathbb{Z}/\ell\mathbb{Z})\) of the form
\[
\begin{pmatrix}
A_1 & 0 & 0 \\
A_2 & A_1 & 0 \\
\vdots & & \ddots \\
0 & \cdots & A_{(p-1)(s-1)/a}
\end{pmatrix},
\]
where \( A_t \) is an \( a \times a \) matrix representing an \( a \)-dimensional irreducible subrepresentation of \( \tau \) on \( \text{Jac}(Y_s)[\ell] \).

Using the bijection between orbits of \( \text{Jac}(Y_s)[\ell] \) and orbits of \( (\mathbb{Z}/\ell \mathbb{Z})\)-covers of \( Y_s \) under \( \tau \) and the above observation for the action of \( \tau \) on \( \text{Jac}(Y_s)[\ell] \), there exists an unramified \( (\mathbb{Z}/\ell \mathbb{Z})^a \)-Galois cover \( \omega : Z_a \to Y_s \) such that \( \psi_a = \phi_a \circ \omega : Z_a \to \mathbb{P}^1_k \) is a Galois cover of \( \mathbb{P}^1_k \) with Galois group of the form \( (\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/\ell \mathbb{Z} \). Also \( \psi_a \) is ramified only over infinity since \( \phi_a \) is ramified only over \( \infty \) and since \( \omega \) is unramified.

### 4 Minimal genus of \( (\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/\ell \mathbb{Z} \)-Galois covers of \( \mathbb{A}^1_k \)

In this section, we find the minimal genus of a curve \( Z \) that admits a covering map \( \psi : Z \to \mathbb{P}^1_k \) ramified only over \( \infty \), with Galois group of the form \( (\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/\ell \mathbb{Z} \). The minimal genus depends only on \( \ell \) and \( p \). We consider the cases \( p \) odd and \( p = 2 \) separately. We also prove that the number of curves \( Z \) of this minimal genus which admit such a covering map is at most \( (p-1)/a \) when \( p \) is odd and at most \( \ell + 1 \) when \( p = 2 \). The following lemma will be useful.

**Lemma 4.1** Let \( G \) be a semi-direct product of the form \( (\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/\ell \mathbb{Z} \) where \( \ell \) is a prime distinct from \( p \). If \( \psi : Z \to \mathbb{P}^1_k \) is a Galois cover ramified only over \( \mathbb{P}^1_k \) with Galois group \( G \), then the subcover \( \omega : Z \to Y \) with Galois group \( (\mathbb{Z}/\ell \mathbb{Z})^b \) is unramified.

**Proof** The quotient of \( G \) by the normal subgroup \( N = (\mathbb{Z}/\ell \mathbb{Z})^b \) is \( \mathbb{Z}/\ell \mathbb{Z} \). Thus the cover \( \psi \) is a composition \( \psi = \phi \circ \omega \) where \( \phi : Y \to \mathbb{P}^1_k \) has Galois group \( \mathbb{Z}/\ell \mathbb{Z} \) and is totally ramified at the unique point \( P_\infty \) over \( \infty \) and where \( \omega : Z \to Y \) has Galois group \( N \) and is branched only over \( P_\infty \). Then \( \omega \) is a prime-to-\( p \) abelian cover of \( Y \). Let \( g \) be the genus of \( Y \). Then by [1, XIII. Cor. 2.12], the prime-to-\( p \) fundamental group of \( Y - \{ P_\infty \} \) is isomorphic to the prime-to-\( p \) quotient \( \Gamma \) of the free group on generators \( \{ a_1, b_1, \ldots, a_g, b_g, c \} \) subject to the relation \( \prod_{i=1}^{b_g} [a_i, b_i] = c^{-1} \). The cover \( \omega \) corresponds to a surjection of \( \Gamma \) onto \( N \) where \( c \) maps to the canonical generator of inertia \( \gamma \) of a point \( \overline{Z} \) over \( P_\infty \). Thus \( N \) is generated by elements \( \{ \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma \} \) subject to the relation \( \prod_{i=1}^{b_g} [\alpha_i, \beta_i] = \gamma^{-1} \). Then \( \gamma = 1 \) since \( N \) is abelian and so \( \omega \) is unramified.

**Theorem 4.2** Let \( p \) be an odd prime. Let \( \ell \) be a prime distinct from \( p \) and let \( a \) be the order of \( \ell \) modulo \( p \). Then:

1. There exists a Galois cover \( \psi_a : Z_a \to \mathbb{P}^1_k \) ramified only over \( \infty \) whose Galois group is a semi-direct product of the form \( (\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/\ell \mathbb{Z} \) such that \( g_{Z_a} = 1 + \ell^a(p-3)/2 \).
2. The integer \( g_{Z_a} \) is the minimal genus of a curve \( Z \) which admits a covering map \( \psi : Z \to \mathbb{P}^1_k \) ramified only over \( \infty \) with Galois group of the form \( (\mathbb{Z}/\ell \mathbb{Z})^b \rtimes \mathbb{Z}/\ell \mathbb{Z} \) for any positive integer \( b \).
3. There are at most \( (p-1)/a \) isomorphism classes of curves \( Z \) which admit a Galois covering map as in part (1) with minimal genus \( g_{Z_a} \).

**Proof** By the construction in Proposition 3.4, there exists a Galois cover \( \psi_a : Z_a \to \mathbb{P}^1_k \) ramified only over \( \infty \) whose Galois group is a semi-direct product of the form \( (\mathbb{Z}/\ell \mathbb{Z})^a \rtimes \mathbb{Z}/\ell \mathbb{Z} \). We compute the genus of the curve \( Z_a \). Recall that \( \psi_a \) is a composition \( \psi = \phi_a \circ \omega \) where \( \omega : Z \to Y_2 \) is an unramified \( (\mathbb{Z}/\ell \mathbb{Z})^a \)-Galois cover and \( \phi_a : Y_2 \to \mathbb{P}^1_k \) has Artin-Schreier equation \( y^p - y = x^2 \). Then \( Y_2 \) has genus \( g_{Y_2} = (p-1)/2 \) by Lemma 3.2(3). By the Riemann-Hurwitz formula, \( 2g_{Z_a} - 2 = \ell^a(2g_{Y_2} - 2) = \ell^a(p-3) \), i.e., \( g_{Z_a} = 1 + \ell^a(p-3)/2 \). This completes part (1).
For part (2), suppose \( \psi : Z \to \mathbb{P}^1_k \) is a Galois cover ramified only over \( \infty \) with Galois group of the form \((\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}\). If \( g \) is the genus of \( Z \), we will show that \( g \geq g_{Z_a} \).

As described in the proof of Lemma 4.1, the cover \( \psi \) is a composition \( \psi = \phi \circ \omega \) where \( \phi : Y \to \mathbb{P}^1_k \) has Galois group \( \mathbb{Z}/p\mathbb{Z} \) and is ramified only over \( \infty \) and where \( \omega \) is unramified with group \((\mathbb{Z}/\ell\mathbb{Z})^b\). By the Riemann-Hurwitz formula, \( 2g - 2 = \ell^b(2g_Y - 2) \).

By Artin-Schreier theory, \( \phi \) is given by an equation \( y^p - y = f(x) \) where \( f \in k[x] \) has degree \( s \) for some integer \( s \) relatively prime to \( p \). Since the genus \( g_Y \) of \( Y \) is \( (p-1)(p-1)/2 \) by Lemma 3.2 (3), we should make \( s \) as small as possible. The value \( s = 1 \) is impossible since \( Y \) is a projective line and there do not exist Galois covers of the projective line ramified only over one point with Galois group \( \mathbb{Z}/\ell\mathbb{Z} \). Thus \( s = 2 \) yields the smallest possible value for \( g_Y \), namely \( (p-1)/2 \). Recall that \( b \geq a \) by Lemma 2.4. Thus \( g \geq 1 + \ell^a(p-3)/2 = g_{Z_a} \).

For part (3), suppose \( \psi : Z \to \mathbb{P}^1_k \) is a Galois cover ramified only over \( \infty \) with Galois group of the form \( (\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z} \) and the genus of \( Z \) satisfies \( g_Z = 1 + \ell^a(p-3)/2 \). As in part (2), \( \psi \) factors as \( \omega : Z \to Y \) is an unramified \((\mathbb{Z}/\ell\mathbb{Z})^a\)-Galois cover, where \( \phi : Y \to \mathbb{P}^1_k \) is an Artin-Schreier cover ramified only over \( \infty \), and where \( Y \) has genus \( (p-1)/2 \). By Lemma 3.2(3), \( Y \) has an affine equation \( y^p - y = a_2x^2 + a_1x + a_0 \) for some \( a_0, a_1, a_2 \in k \) and \( a_2 \notin k^* \). Since \( p \) is odd and \( k \) is algebraically closed, it is possible to complete the square and write \( a_2x^2 + a_1x + a_0 = x_1^2 + \epsilon \) where \( x_1 = \sqrt{a_2}x + a_1/2\sqrt{a_2} \). After modifying by an automorphism of the projective line, specifically by the affine linear transformation \( x \mapsto x_1 \), the equation for \( Y \) can be rewritten as \( y^p - y = x_1^2 + \epsilon \). Since \( k \) is algebraically closed, there exists \( \delta \in k \) such that \( \delta^p - \delta = \epsilon \). Let \( y_1 = y - \delta \). After the change of variables \( y \to y_1 \), the curve \( Y \) is isomorphic to the curve \( Y_2 \) with affine equation \( y_1^p - y_1 = x_1^2 \). Thus there is a unique possibility for the isomorphism class of the curve \( Y \).

From the proof of Proposition 3.4, there is a bijection between \( \tau \)-invariant connected unramified \((\mathbb{Z}/\ell\mathbb{Z})^a\)-Galois covers of \( Y_2 \) and orbits of \( \tau \) on points \( D \) of order \( \ell \) on \( \text{Jac}(Y_2) \). The action of \( \tau \) on \( \text{Jac}(Y_2)[\ell] \) decomposes into \((p-1)/a \) irreducible subrepresentations. Each of these is distinct, because the irreducible factors of \( \Phi_p(t) \in (\mathbb{Z}/\ell\mathbb{Z})[t] \) are distinct. Thus there are \((p-1)/a \) choices for a \( \tau \)-invariant unramified \((\mathbb{Z}/\ell\mathbb{Z})^a\)-Galois cover of \( Y_2 \). Thus there are at most \((p-1)/a \) isomorphism classes of curves \( Z \) which admit a Galois covering map as in part (1) with minimal genus \( g_{Z_a} \).

We note that the set of curves which are unramified \((\mathbb{Z}/\ell\mathbb{Z})^a\)-Galois covers of \( Y_2 \) may contain fewer than \((p-1)/a \) isomorphism classes of curves.

**Theorem 4.3** Let \( p = 2 \) and let \( \ell \) be an odd prime. Then:

1. There exists a Galois cover \( \psi : Z \to \mathbb{P}^1_k \) ramified only over \( \infty \) with Galois group of the form \( \mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \).
2. The minimal genus of a curve \( Z \) which admits a covering map as in part (1) is \( g_Z = 1 \).
3. There are at most \( \ell + 1 \) isomorphism classes of curves \( Z \) which admit a Galois covering map as in part (1) with minimal genus \( g_{Z_a} = 1 \).

**Proof** Note that the order of \( \ell \) modulo 2 is \( a = 1 \). For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form \( \mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \) which is quasi-2. The result is then immediate from Theorem 2.3.

Suppose \( \psi : Z \to \mathbb{P}^1_k \) is a Galois cover ramified only over \( \infty \) with Galois group as in part (1). As before, \( \psi \) factors as a composition \( \phi \circ \omega \), where \( \omega : Z \to Y \) has Galois group \( \mathbb{Z}/\ell\mathbb{Z} \) and \( \phi : Y \to \mathbb{P}^1_k \) is an Artin-Schreier extension with affine equation \( y^2 - y = f(x) \) for some \( f(x) \in k[x] \) of odd degree \( s \). By Lemma 4.1, \( \omega \) is unramified.
The minimal genus for $Z$ will thus occur when $s$ is as small as possible. As before, $s = 1$ is impossible, and so $s = 3$ is the smallest choice. In this case, by Lemma 3.2(3), $gy = 1$, i.e., $Y$ is an elliptic curve. By the Riemann-Hurwitz formula, the minimal genus for $Z$ is $g_Z = 1 + \ell(g_Y - 1) = 1$, which completes part (2).

For part (3), since $k$ is algebraically closed, we can complete the cube of $f(x)$ and make the corresponding change of variables, which is a scaling and translation of $x$. So we can assume that $Y$ has affine equation $y^2 - y = x^3 + a_1x + a_0$ for some $a_0, a_1 \in k$. Then it follows from [11, Appendix A, Prop. 1.1c] that the $j$-invariant of $Y$ is $j(Y) = 0$ and that the discriminant is $\Delta(Y) = (-1)^3 = 1$. Since $k$ is algebraically closed, by [11, Appendix A, Prop. 1.2b], all elliptic curves $Y$ with $j(Y) = 0$ are isomorphic over $k$. Thus there is a unique choice for $Y$ up to isomorphism. Without loss of generality, we may assume that $Y = Y_3$ has affine equation $y^2 - y = x^3$.

From the proof of Proposition 3.4, the action of $\tau$ on $\text{Jac}(Y_3)[\ell]$ decomposes into the direct sum of two 1-dimensional subrepresentations. In other words, the action of $\tau$ is diagonal with both eigenvalues equal to $-1$. The number of non-trivial $\tau$-invariant subgroups of $\text{Jac}(Y_3)[\ell]$ is the number of subgroups of order $\ell$ in $(\mathbb{Z}/\ell\mathbb{Z})^2$, which is $\ell + 1$. As in Theorem 4.2, this implies that there are at most $\ell + 1$ isomorphism classes of curves $Z$ which admit a Galois covering map as in part (1) with minimal genus $g_Z = 1$.

We note that the set of curves which are unramified $\mathbb{Z}/\ell\mathbb{Z}$-Galois covers of $Y_3$ may contain fewer than $\ell + 1$ isomorphism classes of curves.

References


