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The Applications of Mathematics in Finance Actuarial Exam FM Preparation

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Merrimack College-Honors Senior Capstone Project

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Introduction

Background:

Financial mathematics is a growing field in which mathematicians and business professionals alike are regularly finding new links between the two industries. By further developing this area of study, businesses will be able to evaluate trends analytically by implementing mathematical techniques in their investigations. Businesses with a particular interest in financial mathematics include investment banks, commercial banks, hedge funds, insurance companies, corporate treasuries, and regulatory agencies. In particular, the field of actuarial science focuses directly on the applications of mathematics in finance, and requires all aspiring actuaries seeking certification to successfully complete an exam specifically designed to test the applicant's ability to decipher through a series of questions related to the study of financial mathematics[1].

History of Financial Mathematics:

For centuries, economists have used mathematics to create models and theories, but beginning in the late nineteenth century, scientists began to use mathematical techniques and reasoning to explain financial instruments and trends. The founding father of financial mathematics is Louis Bachelier, who used Brownian motion to explain stock prices and mathematical models to analyze financial markets. After the creation of several models for Brownian motion, stochastic calculus was developed by Kiyoshi Itô in 1940 and used to explain the price of financial products in 1969 due to the innovations of Robert Merton. Merton, alongside Fisher Black and Myron Scholes, developed the famous Black-Scholes Model in 1973, which was used to calculate the theoretical price of European put and call options, ignoring any dividends. These innovations laid the ground work for the study of financial mathematics and contributed greatly to actuarial science.

While Louis Bachelier was leading the charge in the implementation of mathematics in the finance field, the Actuarial Society of America was formed in 1889. The first actuary in North America was Jacob Shoemaker of Philadelphia. Since then, the Actuarial Society of America has grown immensely and has become the Society of Actuaries. In addition, another branch of actuarial science focused specifically on property and casualty insurance emerged, the Casualty Actuarial Society. Actuaries use mathematics, statistics, and financial theory to assess and analyze the financial cost of risk[2].

[1] "What Is Financial Math." NCSU Financial Math. 06 July 2014. Web. 02 Aug. 2016. <https://financial.math.ncsu.edu/what-is-financial-math/>.

[2] Skantzos, Nicholas. "Some Background History on Financial Mathematics." 2010. Web. 2 Aug. 2016. <http://itf.fys.kuleuven.be/~nikos/papers/lect2_blackscholes.pdf>

Objective and Methodology:

The objective of this honors senior capstone project is to research the ways that mathematics is used in the field of finance with an emphasis on the techniques applied in the actuarial science industry. This project will also assist those preparing for the second actuarial exam[3].

[3] "Exam FM – Financial Mathematics." Member. Web. 02 Aug. 2016. <https://www.soa.org/education/exam-req/edu-exam-fm-detail.aspx>.

Section 1: Interest Theory

In every instance where money is either borrowed or lent, there will be interest involved. A lender will receive interest since he/she will expect an additional return in exchange for loaning money. A borrower will pay interest since he/she is using the lenders money in place of their own. This section will include a description and examples of the various concepts associated with interest theory.

a) *Simple Interest*

As implied by the name, simple interest is the most straight forward interest to calculate. For the purpose of this paper, we will use r to denote the annual interest rate, P to denote the principal amount or present value, t to denote time, and A to denote the amount after interest is applied or the future value of the investment. The formula for simple interest is:

$$A = P(1 + rt).$$

Example 1.1:

If a bank is paying 2% annual interest for a savings account, and we invest \$500, after three years, what will our account balance be?

Solution:

$$A = 500\big(1 + 0.02(3)\big) = \$530.$$

b) *Compound Interest*

The next type of interest we will discuss is compound interest which accrues interest much faster than simple interest. With compound interest, we earn interest on our interest. That is, interest is earned on the previous amount, not just the principal amount of the investment or loan. The formula for compound interest is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where n is the number of compounding periods per year. The more compounding periods per year the more interest is accrued. That being said, continuous compounding will accumulate the most interest. The formula for continuous compound interest is:

$$A = Pe^{rt}.$$

Example 1.2:

If we were to invest \$1,000 into a savings account earning an interest rate of 3% compounded quarterly, what would the value of our investment be in five years?

Solution:

$$A = 1000\left(1 + \frac{0.03}{4}\right)^{4(5)} = \$1{,}161.18$$

Example 1.3:

Jessica loaned \$200 to Megan at an interest rate of 1.5% compounded continuously with a maturity of three months. How much will Megan need to pay to Jess in three months?

Solution:

$$A = 200e^{0.015(0.25)} = \$200.75$$

c) *Effective Rate*

Most investments and loans are quoted using the nominal rate which is the periodic interest rate multiplied by the number of periods per year. However, the nominal rate does not take into account the compounding that typically occurs. So, we use the effective rate as a more accurate measure of the amount of interest that is charged or accrued. The effective rate can also be used to accurately compare different investment options. If interest is compounded over *n* periods, the formula for effective interest rate, r_e , is:

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

If interest is continuously compounded, the effective interest rate is given by $r_e = e^r - 1$.

Example 1.4:

Michaela has the option of investing in an account that provides an interest rate of 2% compounded monthly or an account with an interest rate of 2.5% compounded quarterly. Which account would give Michaela a higher return on her investment?

Solution:

$$r_e = \left(1 + \frac{0.02}{12}\right)^{12} - 1 = 0.02018 = 2.018\%$$

$$r_e = \left(1 + \frac{0.025}{4}\right)^4 - 1 = 0.025235 = 2.535\%$$

Thus, Michaela is better off investing in the account offering a 2.5% interest rate compounded quarterly.

d) *Inflation and Real Rate of Interest*

Inflation is the general increase in prices which is most often a sign of a growing economy. However, inflation erodes investments, and investors need to carefully construct strategies that account for inflation. Those saving for activities far into the future, such as retirement or college tuition are at a particularly high risk for inflation. The real rate of interest is the rate that accounts for inflation. The real rate is found simply by subtracting the predicted inflation rate from the nominal rate of the investment or loan. On average, the long term inflation rate in the United States over the past one hundred years is 3.22 percent. This means that prices will double approximately every twenty years[4].

Example 1.5:

Bill and Mary are recently married and looking to start planning for their retirement. They are both 25 years of age, plan to retire at age 65 and have found a money market account with an interest rate of 7%. Bill and Mary plan to withdraw $2,500 per month for 30 years upon retirement. How much do Bill and Mary need to invest monthly to meet their retirement goals?

Solution:

First we will calculate the present value of the monthly deposits

$$PV = x\sum_{i=1}^{480}\left(1 + \frac{0.07}{12}\right)^{-i} = x(160.91884)$$

where *x* is the monthly deposit.

Next we will calculate the present value of the monthly withdrawals

$$PV = 2500\sum_{i=481}^{841}\left(1 + \frac{0.07}{12}\right)^{-i} = 23{,}055.9537$$

Set these results equal to each other and solve for *x*

$$160.91884x = 23055.9537$$
$$x = \$143.277$$

[4] McMahon, Tim. "InflationData.com." *US Inflation Long Term Average*. 01 Apr. 2014. Web. 15 Sept. 2016. <http://inflationdata.com/Inflation/Inflation_Rate/Long_Term_Inflation.asp>

e) Discount Rate

The discount rate is often used interchangeably with the interest rate. However, the discount rate refers the interest rate we would need to earn on a given amount of money today to end up with a given amount in the future.[5] The discount rate can also be defined as the rate used in discounted cash flow analysis. Discounted cash flow analysis is the calculation of the present value of future cash flows. In practical uses, some financial securities are sold below the face value to entice buyers to invest. When a financial instrument is sold below the face value it is classified as below par. The difference between the face value and the price of the security divided by the face value is the discount factor, which is measured as a percent. The discount rate is the rate at which the selling price would accumulate to the face value in a specified time period. Pure discount, or zero coupon bonds, are typical examples of this phenomenon. The discount rate of a pure discount bond can be found with the formula:

$$P = \frac{V}{(1+r)^t}$$

Where *P* is the price at which the bond is selling and *V* is the face value.

Example 1.6:

Suppose a pure discount bond is being traded for $96.50 and has a face value of $100. If the bond has a maturity of five years, what is the discount rate?

Solution:

$$96.50 = \frac{100}{(1+r)^5}$$

$$\left(\frac{100}{96.5}\right)^{1/5} - 1 = r = 0.00715 = 0.715\%$$

f) Time Value of Money

The fundamental idea of the time value of money is that a dollar today is not worth a dollar tomorrow. The same dollar today is actually worth less in the future. The first reason for this is that consumers prefer present consumption to future consumption. In order to encourage future consumption, consumers will need to be offered more at a future point in time. Thus, as the desire for present consumption increases, the value of

[5] Picardo, CFA Elvis. "Discount Rate." *Investopedia*. 23 Jan. 2014. Web. 15 Sept. 2016. <http://www.investopedia.com/terms/d/discountrate.asp>.

the dollar will decrease. Another reason for the decrease in future dollar valuation is inflation, which is the general rise in prices. As prices increase, the purchasing power of one dollar decreases. Lastly, if there is any uncertainty or risk associated with the future cash flow, the future cash flow will have a lower value[6].

g) *Future Value*

Relating to the time value of money, the future value is the value of a current asset at a specific time in the future given an interest rate. The future value is used to compare the profit of various investments into the future. Since the future is unknown, some future value calculations are not as easy to determine. For example, stocks are highly volatile and dividend payments can change at varying rates. However, bonds, annuities and other fixed securities can be easily compared using a future value calculation. The formula used to determine the future value, *FV*, of a lump sum with compounding is:

$FV = P\left(1+\frac{r}{n}\right)^{nt}$ or $FV = Pe^{rt}$ for continuous compounding.

Example 1.7:

Calculate the value of a lump sum of $500 deposited today in a savings account that earns 1.25% interest compounded monthly in ten years.

Solution:

$$FV = 500\left(1+\frac{0.0125}{12}\right)^{12*10}$$

$$FV = \$556.53$$

h) *Present Value*

The opposite of future value, is present value. The present value represents the value of a future stream of payments or sum of money given a specific interest rate. The interest rate at which the money is discounted is called the discount rate. Similar to future value, the present value can be used to accurately compare investment or loan options. The known future values are discounted at the discount rate back to time zero and the higher the discount rate, the lower the present value. The formula for the present value of a lump sum is:

$P = A\left(1+\frac{r}{n}\right)^{-nt}$ or $P = Ae^{-rt}$ if compounding continuously.

[6] Damoda, Aswath. *The Time Value of Money*. Web. 20 Sept. 2016. <http://people.stern.nyu.edu/adamodar/pdfiles/acf4E/presentations/timevalue.pdf>.

Similarly, the present value of a stream of income is given by:

$$P = \int_0^T P(t)e^{-rt}dt$$

Where $P(t)$ is income stream at each value of *t*.

Example 1.8:

Olivia is deciding between two investments that will make annual payments for the next three years. The first option will pay \$100 in year one, \$150 in year two, and \$125 in year three. The other investment will pay \$125 in year one, \$100 in year two, and \$150 in year three. Assuming an interest rate of 3% compounded quarterly, which investment should Olivia choose?

Solution:

Option 1:

$$PV = 100\left(1+\frac{0.03}{4}\right)^{-4(1)} + 150\left(1+\frac{0.03}{4}\right)^{-4(2)} + 125\left(1+\frac{0.03}{4}\right)^{-4(3)} = \$352.63$$

Option 2:

$$PV = 125\left(1+\frac{0.03}{4}\right)^{-4(1)} + 100\left(1+\frac{0.03}{4}\right)^{-4(2)} + 150\left(1+\frac{0.03}{4}\right)^{-4(3)} = \$352.65$$

Thus, although fairly insignificant in this example, Olivia should invest in the second option.

i) *Net Present Value*

The net present value, often abbreviated as NPV, is the difference between the present value of cash inflows and the present value of cash outflows. Net present value is often used in budgeting to determine the profitability of a project or investment. The formula for net present value is:

$$NPV = \sum_{i=1}^{T} \frac{C_i}{(1+r)^i} - C_o$$

Where C_i is the cash flow, and C_0 is the initial cash outflow or investment.

Example 1.9:

ABC Corporation is considering investing in building a new structure for their manufacturing operations. The initial investment for the project is $1,000. Over the next five years, the company expects to take in $150, $160, $210, $250, and $275 each year. With an interest rate of 3%, will this project be profitable?

Solution:

$$NPV = \left(\frac{150}{(1+0.03)^1} + \frac{160}{(1+0.03)^2} + \frac{210}{(1+0.03)^3} + \frac{250}{(1+0.03)^4} + \frac{275}{(1+0.03)^5}\right) - 1000$$

$$NPV = 947.965 - 1000 = -\$52.03$$

Thus, this investment would not be profitable and ABC Corporation should not embark on this project.

Section 2: Annuities

Annuities are financial products sold by financial institutions and can be used to generate a steady stream of cash flow for some future amount of time. Individuals typically make deposits into an account up until a certain date and then receive payments for a specified number of periods. These payments are typically equal, or level, but they can also vary, known as a non-level annuity. Some examples of situations in which individuals would use annuities are retirement accounts, paying off loans, life insurance, and pension funds.

a) Annuity-immediate

An annuity-immediate or annuity due is an annuity in which payments are made at the end of each time period. Typically individuals are interested in the present value of their future cash flow. The present value of the annuity is simply the sum of the present values of each payment. This can be calculated using the following formula:[7]

$$PV = P\left(\frac{1-(1+r)^{-n}}{r}\right)$$

Where *P* is the payment amount, *r* is the interest rate per payment period, and *n* is the number of payments.

Example 2.1:

Calculate the present value of an ordinary annuity that makes payments of $1,000 annually for 10 years at an interest rate of 2%.

Solution:

Using the formula, with $P = 1{,}000$, $r = 0.02$, and $n = 10$, we find that the present value is $PV = 1000\left(\frac{1-(1+0.02)^{-10}}{0.02}\right) = \$8{,}982.585$.

b) Annuity due

An annuity due is similar to an ordinary annuity except the payment is received at the beginning of each time period as opposed to the end. This type of annuity is not as common but is another viable investment option that provides a steady stream of income. A common example of an annuity due is rent payments received by a landlord. An annuity due is more beneficial for the recipient because it provides them with a higher cash flow faster which can be invested for a longer period of time. On the other hand, individuals paying the annuity miss out on this benefit since they are obligated to make the payment sooner. The type of

[7] Finan, Marcel B. "A Basic Course in the Theory of Interest and Derivatives Market." 157. Arkansas Tech University, 15 Aug. 2015. Web. 30 Sept. 2016. <http://faculty.atu.edu/mfinan/actuarieshall/mainf.pdf>.

annuity is agreed upon in the contract between the recipient and beneficiary. Similar to ordinary annuities, it is often desirable to compare various options for annuity dues which can be accomplished by comparing the present values of the streams of income. The present value can be calculated with the following formula:[8]

$$PV = P + P\left(\frac{1-(1+r)^{-(n-1)}}{r}\right)$$

Where P is the payment paid or received, r is the interest rate, and n is the number of time periods.

Example 2.2:

Calculate the present value of an annuity due that makes payments of $1,000 annually for 10 years at an interest rate of 2%.

Solution:

Using the formula with $P = 1{,}000$, $r = 0.02$, and $n = 10$, we find that the present value is $PV = 1{,}000 + 1{,}000\left(\frac{1-(1+0.02)^{-(10-1)}}{0.02}\right) = \$9{,}162.237$. This annuity due has a higher present value than the ordinary annuity calculated above.

c) *Perpetuity*

A perpetuity is a type of annuity that has an infinite term. Similar to annuities, perpetuities can be due or received at the beginning or end of each time period, as specified in a contract. Since perpetuities are infinite, there is no way to calculate the accumulated value of the security, but we can calculate the present value by using the concept of geometric series. The formula for calculating the present value of an ordinary perpetuity (payment due or received at the end of the time period) is:

$$PV = \frac{P}{r}$$

Where P is the payment due or received and r is the discount rate. Similarly, the formula for the present value of a perpetuity due (payment due or received at the beginning of the time period) is:

$$PV = \frac{P}{r} + P_0$$

Where P_0 is the initial payment[9].

[8] Root. "Annuity Due." *Investopedia*. 28 July 2015. Web. 1 Oct. 2016. <http://www.investopedia.com/terms/a/annuitydue.asp>.

[9] "PreMBA Finance." *PreMBA Finance*. Web. 1 Oct. 2016. <http://ci.columbia.edu/ci/premba_test/c0332/s5/s5_4.html>

Example 2.3:

Calculate the present value of the ordinary perpetuity that makes payments of $1,000 with an interest rate of 2%. Then calculate the present value of a perpetuity due with the same payment and interest rate.

Solution:

To calculate the present value of the ordinary perpetuity, we use the formula $PV = \frac{P}{r} = \frac{1{,}000}{0.02} = \$50{,}000$. To calculate the present value of the perpetuity due, we use the formula $PV = \frac{P}{r} + P_0 = \frac{1{,}000}{0.02} + 1{,}000 = \$51{,}000$.

d) Payable monthly or continuously

In some instances, annuities and perpetuities will be compounded more often than just annually. When this occurs, we must calculate the effective interest rate before implementing any of the formulas we have used above. The formula to calculate the effective interest rate is:

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

Where *r* is the annual interest rate, *n* is the number of compounding periods per year, and *t* is the number of years[10]. From here, we will use the present value formula for the typical annuity or perpetuity as previously stated.

Example 2.4:

Find the present value of an ordinary annuity of $1,000 monthly for 2 years with an interest rate of 2% compounded quarterly.

Solution:

First, we must find the effective interest rate with $r = 0.02$, $n = 4$, and $t = 2$.

$$r_e = \left(1 + \frac{0.02}{4}\right)^4 - 1 = 0.020151$$

Next, we will use this information to calculate the present value of an ordinary annuity with $P = 1{,}000$, $n = 2(12) = 24$, and $r = 0.020151$.

$$PV = 1{,}000\left(\frac{1 - (1 + 0.020151)^{-24}}{0.020151}\right) = \$18{,}881.612$$

[10] "Financial Mathematics for Actuaries." Web. 5 Oct. 2016. <http://www.mysmu.edu/faculty/yktse/FMA/S_FMA_2.pdf+>.

e) Non-level annuities/cash flows

So far, we have discussed level annuities and perpetuities that have the same payment or cash flow for each time period. However, there are some contracts that have varying payment amount which are known as non-level annuities. Two types of non-level annuities are arithmetic and geometric. Arithmetic annuities have a common difference between successive payments and geometric annuities have a common ratio.

f) Geometric increasing/decreasing annuity

A geometric annuity is an annuity that that has payments that create a geometric progression. Instead of having a common difference as in an arithmetic annuity, a geometric annuity has a common multiple. For example, an annuity may have a growth rate of 10% each year so each successive payment would be 10% more than the last. If the common multiple is positive, the annuity is increasing whereas if the common multiple is negative, the annuity is decreasing. The formula for calculating the present value of a geometric annuity is:

$$PV = \frac{P}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right]$$

Where *P* is the first payment, *r* is the interest rate, *g* is the growth rate, and *n* is the number of periods[11].

Example 2.6:

Calculate the present value of the an annuity that has payments of $100, $110, $121, $133.10, and $146.41 that has an interest rate of 2%.

Solution:

First we must determine whether or not this annuity is geometric by attempting to calculate the common multiple between the payments. So, $\frac{110}{100} = 1.1,\ \frac{121}{110} = 1.1, \frac{133.10}{121} = 1.1, etc.$ Therefore, the common multiple is 1.1 or 10%. Using the above formula, the present value of the annuity is:

$$PV = \frac{100}{0.02-0.10}\left[1-\left(\frac{1+0.10}{1+0.02}\right)^5\right] = \$440.753$$

g) Term of annuity

The number of periods of an annuity or the length of time of the annuity is referred to as the term of the annuity. It would be useful to find the term of the annuity when calculating how

[11] "Present Value of a Growing Annuity." *Finance Formulas*. Web. 15 Oct. 2016. <http://www.financeformulas.net/Present_Value_of_Growing_Annuity.html>.

long it will take to pay off a debt with a certain regular payment. The equation for solving for the term of the annuity is:

$$n = ln\left[\left(1 - \frac{PV(r)}{P}\right)^{-1}\right] \div ln(1 + r)$$

Where PV is the present value of the annuity, P is the payment or cash flow amount, and r is the interest rate[12].

Example 2.7:

Bob is purchasing a car and taking out a loan for $5,000 at a rate of 4% and plans to make monthly payments of $250. How many payments will Bob need to make to pay off his debt?

Solution:

Using the above formula for n, where $PV = 5{,}000$, $r = 0.04$, and $P = 250$ we can solve for the number of payments John will need to make.

$$n = ln\left[\left(1 - \frac{5000(0.04)}{250}\right)^{-1}\right] \div ln(1 + .04) = 41.035\, payments$$

[12] "Annuity-Solve for Number of Periods." *Finance Formulas*. Web. 20 Oct. 2016. <http://www.financeformulas.net/Number-of-Periods-of-Annuity-from-Present-Value.html>.

Section 3: Loans

Borrowing money is vital in many aspects of life and it is important to know how to calculate the interest earned on loans, the amount of time it will take to pay off the loan, and the different options there are to repay the debt. Depending on which side of the loan you are on, the borrower or lender, you will have different objectives.

a) Principal, Interest, and Term of a loan

The principal of the loan is the total amount borrowed or lent. Interest accumulates on this amount and the terms of interest are agreed upon at the time of the development of the loan. There is an explicit interest rate that will be paid or received and depends on the individual's creditworthiness. The term of the loan refers to the length of the loan. There is always a deadline for the loan to be paid off and the monthly payments are typically derived from the time of the loan and interest rate. In order to decrease the risk of default, frequently there is some sort of collateral for the loan. For example, mortgage and car loans are backed by the house and car respectively. Since there is collateral, the loan is deemed less risky than a loan with no security and as such would have a lower interest rate.

b) Final payment (drop payment, balloon payment)

The final payment of a loan can either be greater than, less than, or equal to the previous payments made on the loan. If the payment is greater than the typical payments, it is said to be a balloon payment whereas if it is less than, it is a drop payment. The formula for calculating the drop payment amount is:

$$P_d = P\left(\frac{(1+r)^t - 1}{r}\right)(1+r)^{1-t}$$

Where P is the typical payment amount, r is the interest rate, and t is the time at which the payment is made with regards to the term of the loan. The formula for calculating the balloon payment is[13]:

$$P_b = P + P\left(\frac{(1+r)^t - 1}{r}\right)(1+r)^{1-t}$$

Example 3.2:

Calculate the balloon payment of John's loan with a principal of $20,000 if he has been making monthly payments of $1,000, the term of the loan is two years, and the monthly interest rate is 1%. John is making this payment on the twenty third month of his loan.

[13] "Mathematical Interest Theory." *Google Books*. Web. 20 Oct. 2016.

Solution:

Using the formula for determining the balloon payment amount,

$$P_b = 1000 + 1000\left(\frac{(1.01)^{23/24} - 1}{0.01}\right)(1.01)^{1-23/24} = \$1958.53$$

c) Amortization

One method to pay off a loan is amortization. With this method, each payment is first used to pay off the interest accrued since the last payment. The remaining portion of the payment is then used to reduce the principal amount of the loan. We can construct an amortization table that calculates the amount of interest and principal that is paid off at each payment period. To construct an amortization table, we will need five columns including the year, payment, interest payment, principal payment, and outstanding balance. The following chart is an example of what an amortization schedule would look like where *X* is the principal of the loan, *r* is the interest rate, *P* is the payment amount, and *Y* is the remaining balance after *n* years.

Year	Payment	Interest Payment	Principal Payment	Outstanding Balance
				$X
1	P	r*X	P-(r*x)	X-(P-(r*X))
...	...	...	...	...
n	P	0	$Y	0

Example 3.3:

Create an amortization schedule for a car loan of $2,000 that requires monthly payments of $500 with an interest rate of 8%. How many years will it take to pay off the loan?

Solution:

The amortization schedule is given by:

Year	Payment	Interest Payment	Principal Payment	Outstanding Balance
				2000
1	500	160	340	1660
2	500	132.8	367.2	1292.8
3	500	103.424	396.576	896.224
4	500	71.69792	428.30208	467.92192
5	500	37.4337536	462.5662464	5.3556736
6	500	0.428453888	499.5715461	-494.2158725

Therefore, the loan will be paid off between 5 and 6 years.

d) Sinking fund

Another method to calculate the repayment schedule of a loan is through a sinking fund. The borrower may choose to make deposits into a fund known as a sinking fund to accumulate the principal amount of the loan plus interest. Therefore, at the end of the term of the loan, the borrower will be able to make a lump sum payment to completely repay the principal amount.

There are two interest rates that we must be aware of when using the sinking fund method. There is the interest rate of the loan and the interest rate that the sinking fund provides. Thus, we must calculate the total we will owe for the loan at maturity including interest before determining the dollar amount of monthly deposits that we must make. To calculate the future value of the loan, we will need to know the interest rate and the time of the loan. Then, we will simply use the future value formula that we discussed in Section 1. Next, we will create a schedule similar to the amortization schedule that shows the accumulated interest on our deposits. The following chart is an example of a sinking fund schedule[14]:

Time	Deposit	Interest Accrued	Current Balance
0	P	0	X
1	P	(1+r)*X	P+((1+r)*X)
...	...	...	...
n	P	(1+r)*Y	P+(1*r)+Y)

Example 3.4:

Jessica wants to pay off her car loan through a sinking fund. The loan is for $4,000 and charges interest of 1.5% compounded monthly. She has found a sinking fund that accrues interest at 1.4% compounded monthly and believes she will be able to make monthly deposits of $500. How long will it take Jessica to pay off this loan?

Solution:

First, we will construct the sinking fund schedule given that Jessica will deposit $500 per month which accrues interest at 1.4% compounded monthly.

[14] "Amortization Method and Sinking Funds." Web. Oct. 2016. 163. <http://www.mheducation.ca/college/zima/graphics/zima5mfz_information/5mfz_sample.pdf+>.

Time	Deposit	Interest Accrued	Current Balance
0	500	0	500
1	500	7	1007
2	500	1019.084	1519.084
3	500	1537.313008	2037.313008
4	500	2061.760764	2561.760764
5	500	2592.501893	3092.501893
6	500	3129.611916	3629.611916
7	500	3673.167259	4173.167259
8	500	4223.245266	4723.245266
9	500	4779.924209	5279.924209

Next, we must calculate the future value of the $1,000 loan compounded monthly at 1.5% using the future value formula $FV = P\left(1+\frac{r}{n}\right)^{nt}$ where *P* is the principal amount, *r* is the rate, *n* is the number of compounding periods, and *t* is the amount of time.

$$FV = 4000\left(1+\frac{0.015}{12}\right)^{12*8} = \$4509.65$$

Section 4: Bonds

Bonds are one of the safest investment options and are great investment alternatives for those seeking an almost guaranteed cash flow. There are several different types of bonds including government, corporate, pure discount, and coupon. Each bond received a rating that measures the amount of risk associated with the bond and with the extra risk of lower rated bonds comes some excess return.

a) *Price, Face Value, and Book Value*

The face value of the bond is the nominal or dollar value of the security. Typically the face value is $1,000. Bonds rarely trade at the face value, which is known as trading at par. Instead they will trade above or below par. If the bond offers coupon payments, it will trade above par, whereas a pure discount, non-coupon paying bond will trade below par. The price at which the bond is traded is known as the price of the bond. Finally, the book value, also referred to as the carrying value of the bond, is the present value of all of the payments that are to be made.

b) *Amortization of premium*

When a bond is issued at premium, meaning the price of the bond is greater than its' inherent face value, the debt is gradually written off through a process of amortization. Similar to the amortization schedule for a loan, the amortization of the premium of the bond is easily computed through the use of a table. We will want columns for the time period, opening price, interest, payment amount, closing price, and the premium[15]. An example of an amortization table for bond premiums is as follows where X is the price of the bond, r is the rate at which the bond accrues interest, i is the bond rate, and Y is the opening value after n time periods:

Period	Opening	Interest	Payment	Closing	Premium
1	X	r*X	i*X	X+(r*X)-(i*X)	X-(X+(r*X)-(i*X))
...	...	...	...	...	...
n	Y	r*Y	i*X	Face Value	Y-Face Value

Example 4.1:

A premium bond is trading for $1,250, has a face value of $1,000, earns an interest rate of 5%, and the bond rate is 7%. Construct an amortization schedule of the premium bond.

15 "Bond Amortization Schedule - Effective Interest Method." *Double Entry Bookkeeping*. 06 Nov. 2016. Web. 15 Nov. 2016. <http://www.double-entry-bookkeeping.com/bonds-payable/bond-amortization-schedule/>.

Solution:

Period	Opening	Interest	Payment	Closing	Premium
1	1250	62.5	87.5	1225	25
2	1225	61.25	87.5	1198.75	26.25
3	1198.75	59.9375	87.5	1171.1875	27.5625
4	1171.188	58.55938	87.5	1142.246875	28.940625
5	1142.247	57.11234	87.5	1111.859219	30.38765625
6	1111.859	55.59296	87.5	1079.95218	31.90703906
7	1079.952	53.99761	87.5	1046.449789	33.50239102
8	1046.45	52.32249	87.5	1011.272278	35.17751057
9	1011.272	50.56361	87.5	974.335892	36.93638609

Thus, the premium will be amortized between the eighth and ninth year of the bond.

c) Redemption value

When the bond reaches it maturity, the owner of the bond will receive the agreed upon redemption value. If the redemption value is equal to the face value it is redeemed at par. Typically the bond will be redeemed above or below par depending on whether the bond was pure discount or paid coupons.

d) Coupon and coupon rate

Some bonds provide equal payments throughout the life of the bond known as coupon payments. The payments can be expressed as a percentage of the face value of the bond as the coupon rate. Depending on the type of bond, coupons may be paid annually, semiannually, quarterly, or monthly. The coupon payment plays a large role in pricing a bond since investors will need to account for the cash flow they receive over the life of the bond in order to determine a fair price. The formula used to calculate the price of a bond is:

$$P = \frac{V}{(1+y)^T} + \sum_{i=1}^{nT} \frac{C_i}{(1+y)^{\frac{i}{n}}}$$

Where C_i denotes the coupon payment at time *i*, *y* is the yield, *n* is the number of times the coupon is paid per year, *V* is the face value, and *T* is the maturity.

Example 4.2:

Calculate the price of a ten year bond that has a face value of $1,000, pays semiannual coupons of $25, and has a yield rate of 3%.

Solution:

Using the above formula for pricing a bond, this bond should be priced at:

$$P = \frac{1000}{(1.03)^{10}} + \sum_{i=1}^{10(2)} \frac{25}{(1.03)^{\frac{i}{2}}} = \$1{,}173.779$$

e) Callable/non-callable

A traditional bond that pays regular coupons or is sold at a discount is also referred to as a non-callable bond. On the other hand, a slightly riskier investment that can offer a higher payoff is a callable bond. A callable bond has two potential expiration dates; the original maturity date and the callable date. At the callable date, the issuer may choose to call the bonds from the investor and is essentially able to retire the bonds before the original maturity date. A bond issue would want to call the bond if interest rates were to drop since the issuer would be able to reissue the bond at the lower rate[16]. For example, if an investor purchases a 10-year callable bond with an interest rate of 3% and a callable date of 2 years, and the interest rate drops to 2% after 2 years, the issuer will call the bond and reissue it at the 2% interest rate.

f) Calculate the yield rate

To calculate the yield rate of a bond, we will use the formula for calculating the price of a bond and simply solve for the rate. The yield of a zero coupon or pure discount bond is given by the equation:

$$y = \left(\frac{V}{PV}\right)^{\frac{1}{n}} - 1$$

Where *V* is the face value, *PV* is the present value, and *n* is the number of periods of the bond. To calculate the yield of a bond that pays coupons, simply use the pricing formula for a coupon bond and solve for the yield.

Example 4.3:

Calculate the yield rate of a ten year pure discount bond that has a face value of $1,000 and is currently trading for $950.

Solution:

The yield for this bond is $y = \left(\frac{1000}{950}\right)^{\frac{1}{10}} - 1 = 0.00514 = 0.514\%$

[16] Petroff, Eric. "Callable Bonds: Leading A Double Life." *Investopedia*. 18 Apr. 2007. Web. 15 Nov. 2016. <http://www.investopedia.com/articles/bonds/07/callable_bonds.asp>.

Section 5: General Cash Flows and Portfolios

Constructing a well thought out, diversified portfolio is essential to optimizing long term return on investment. There are various measures that can be taken to determine which type of investment is best for each individual depending on their risk tolerance and objectives. In addition, there are several methods that can be used to compare investment options including matching Macaulay and modified duration, and relating rates of return.

a) *Dollar-weighted rate of return*

One method that can be used to determine the rate of return of a portfolio is the dollar-weighted or money-weighted rate of return. This rate of return is calculated by equating the present value of all of the cash flows and terminal values equal to the initial value of the investment. This rate is also known as the internal rate of return. The dollar weighted rate of return can be affected heavily by the time at which larger cash flows are received and therefore can provide an accurate measurement of actual dollar amounts invested over time[17]. To calculate the dollar-weighted rate of return, we will set the present value of cash outflows equal to the present value of all cash inflows. We can use the formula:[18]

$$NPV = \sum_{i=1}^{T} \frac{C_i}{(1+y)^i} - C_0$$

Where C_i is the cash flow at time i, C_0 is the initial cash flow, *NPV* is the net present value, and *y* is the yield rate. Then to find the rate of return, r $= \frac{NPV}{C_0}$.

Example 5.1:

Calculate the dollar weighted rate of return of an investment that has an initial investment of $500 and annual cash flows of $150, $125, $175, $150 with an interest rate of 2%.

Solution:

$$NPV = \left(\frac{150}{(1.02)^1} + \frac{125}{(1.02)^2} + \frac{175}{(1.02)^3} + \frac{150}{(1.02)^4}\right) - 500 = 70.688$$

Next, the rate of return is $\frac{NPV}{C_0} = \frac{70.688}{500} = 14.14\%$.

[17] Root. "Money-Weighted Rate Of Return." *Investopedia*. 06 May 2008. Web. 25 Nov. 2016. <http://www.investopedia.com/terms/m/money-weighted-return.asp>.
[18] "Money Vs. Time-Weighted Return - CFA Level 1." *Investopedia*.18 Apr. 2008. Web. 20 Nov. 2016. <http://www.investopedia.com/exam-guide/cfa-level-1/quantitative-methods/discounted-cash-flow-time-weighted-return.asp>.

b) Time-weighted rate of return

Similar to the dollar-weighted rate of return, the time-weighted rate of return is used to measure the rate of return of a portfolio of assets. However, the time-weighted rate of return is the measure of the compound rate of growth of a portfolio. This rate is also known as the geometric mean return. This rate assumes that all cash inflows are reinvested in the portfolio and the rate is converted into an annual rate[19]. The formula for calculating the annualized time-weighted rate of return of a portfolio is:

$$(1 + r_1)^{\frac{t_1}{T}}(1 + r_2)^{\frac{t_2}{T}} \dots (1 + r_i)^{\frac{t_i}{T}} - 1$$

Where r_i is the rate of the cash flows at each time period *i*, *T* is the maturity date, and t_i is the time interval.

Example 5.2:

Calculate the time-weighted rate of return of a set of cash flows that have a rate of 3% for the first two years and 4% for the last three years.

Solution:

Using the above formula, we can easily calculate the annualized time-weighted rate of return

$$r = (1.03)^{\frac{2}{5}}(1.04)^{\frac{3}{5}} - 1 = 3.5989\%$$

c) Duration (Macaulay and modified)

Macaulay duration is the percentage change in price for a 100 basis point change in rates. It measures the sensitivity to changes in prices of a portfolio or single asset. The formula for calculating the Macaulay duration of a coupon bond is:

$$D = \frac{T\frac{V}{(1 + y)^T} + \sum_{i=1}^{nT} \frac{i}{n} \frac{C_i}{(1 + y)^{\frac{i}{n}}}}{P}$$

Where *T* is the maturity, *V* is the face value, *y* is the yield rate, *n* is the number of compounding periods, and C_i is the coupon payment at time *i*. To calculate the duration of a portfolio of assets, we can use the formula:

$$D = \sum_j \alpha_j \frac{P_j}{P} D_j$$

Where α_j is the number of shares of each stock *j*, P_j is the price of stock *j*, P is the value of the portfolio, and D_j is the duration of stock *j*.

[19] Root. "Time-Weighted Rate of Return." *Investopedia*. 11 Oct. 2015. Web. Nov. 2016. <http://www.investopedia.com/terms/t/time-weightedror.asp?lgl=no-infinite>.

In addition to Macaulay duration, modified duration can be used to approximate the percentage change in a bond's price for a 100 basis point change in yield with the assumption that the bond's expected cash flows does not change when the yield changes. Calculating the modified duration of a stock or portfolio depends on the Macaulay duration and can be found using the formula:[20]

$$D = \frac{Macaulay\ Duration}{\left(1 + \frac{y}{n}\right)}$$

Where *y* is the yield rate and *n* is the number of compounding periods.

Example 5.3:

Calculate the Macaulay and modified durations of a two year coupon bond with a face value of $1,000 that pays semi-annual coupons of $25 and has an interest rate of 2% and is currently priced at $1,029.

Solution:

The Macaulay duration can be calculated with the formula:

$$D = \frac{2\frac{1000}{(1.02)^2} + \frac{1}{2}\left(\frac{25}{1.02^{.5}}\right) + \left(\frac{25}{1.02}\right) + \frac{3}{2}\left(\frac{25}{1.02^{1.5}}\right) + 2\left(\frac{25}{1.02^2}\right)}{1029} = 1.986$$

We can now calculate the modified duration:

$$D = \frac{1.986}{\left(1 + \frac{0.02}{2}\right)} = 1.966$$

d) Convexity

Convexity is the measure of how the duration of the bond changes as the interest rate changes. This measure is often used as a risk management tool that helps determine the amount of market risk the bond is exposed to. There is an inverse relationship between bond prices and interest rates. That is, as interest rates rise, bond prices decrease, and vice versa. Convexity is a stronger measure of interest rate risk than duration because unlike duration, it does not assume a linear relationship between bond prices and interest rates. Duration is a good measure for small changes in interest rates but convexity is better for assessing the impact on bond prices when there are large fluctuations in interest rates. As convexity increases, the systematic risk that the portfolio is exposed to increases and vice versa. In

[20] Root. "Modified Duration." *Investopedia*. 29 Dec. 2003. Web. Nov. 2016. <http://www.investopedia.com/terms/m/modifiedduration.asp>.

general, the higher the coupon rate, the lower the convexity (market risk) of the bond. To calculate convexity, we can use the following formula:[21]

$$C = \frac{\frac{1}{(1+i)^2}\left[\sum_{t=1}^{T}\frac{CF_i}{(1+i)^t}(t^2+t)\right]}{P}$$

Where CF_i is the cash flow at time *i*, *t* is the current time, *i* is the interest rate, and *V* is the face value.

Example 5.4:

Calculate the convexity of a one year bond that pays a coupon of $25 semiannually, has an interest rate of 2% and a face value of $1,000.

Solution:

We will use the above formula for convexity to calculate the sensitivity to interest rate changes of this coupon bond.

$$C = \frac{\frac{1}{(1.02)^2}\left[\frac{25}{(1.02)^2}(0.5^2+0.5)+\frac{25}{(1.02)^2}(1^2+1)\right]}{1000} = .0635$$

e) Forward rate

A forward rate is a rate between two time periods that depends on the quoted spot rate for each time interval. The forward rate can be used to calculate future values and the rate that is earned or charged between time periods. The formula used to calculate the forward rate between two time intervals is:

$$\left(1+r_{ty}\right)^t\left(1+f_{ty,uy}\right)^{u-t} = \left(1+r_{uy}\right)^u$$

Where r_{ty} is the spot rate of the bond at time ty, *t* is the time of one bond, *u* is the time of the other bond, and $f_{ty,uy}$ is the forward rate between the two time periods.

Example 5.5:

Calculate the forward rate between year 1 and 2 of a bond that has a one-year spot rate of 2% and a two-year spot rate of 4%.

Solution:

Using the formula that calculates the forward rate:

$$(1.02)^1\left(1+f_{1y,2y}\right)^1 = (1.04)^1$$

Therefore, $f_{1y,2y} = 0.01961 = 1.961$

[21] Root. "Convexity Adjustment." *Investopedia*. 03 Oct. 2012. Web. Nov. 2016. <http://www.investopedia.com/terms/c/convexity-adjustment.asp>.

Section 6: Derivatives and Options

Other financial instruments to include in a portfolio are derivatives and options. A derivative is a financial contract that gets its value from an underlying asset. Options are one specific type of derivative. Other categories include swaps, future contracts, and forward contracts. There are several reasons to choose to include derivatives in a portfolio, and investing in derivatives will further diversify and decrease the assumed risk of investing.

a) Derivatives, underlying assets

A derivative is a contract which specifies the right or obligation to receive or deliver a certain asset for a specified price. The value of the derivative depends on the value of another asset. The asset on which the price of the derivative depends is known as the underlying asset. Investors may choose to enter into the derivative markets for risk management, speculation, to reduce transaction costs, and potential arbitrage opportunities. Derivatives are typically traded on commodities, stock, stock indexes, currency exchange rates, and interest rates[22].

b) Short selling

If an investor is buying an asset, he or she is said to have a long position on the stock. On the other hand, if the investor is selling an asset, he or she has a short position. Another method to use when purchasing stocks is short selling. When selling short, the investor believes that the stock price is going to decrease. This is one method that can be used to make a profit when the stock market is decreasing. To sell short a stock the investor must first borrow x number of shares, which will be returned at a specified date. The investor will then sell the shares at the current stock price S_0. After selling the borrowed shares, the investor waits until the stock price drops at which point he or she will purchase the shares at the new price. Since the stock price dropped, the investor does not need to use all of the proceeds from selling the borrowed stock, and thus has made a profit. At this point, the investor must return the shares to the firm or individual that he or she initially borrowed from. However, if the investor is wrong in his or her assumption that the stock price will fall, he or she will suffer a loss from the short position. If the stock price instead increases, the investor will need to purchase the stock at the higher price in order to return the shares to the lending firm or individual. In this instance, he or she must use the proceeds received from selling the shares and additional money of his or her own to purchase the more expensive stock.

Example 6.1:

Ruthanne sold short 20 shares of Disney stock priced at \$106 with a transaction cost of \$2 per share. She was correct in her bet that the share price would decrease, and she closes her

[22] "Derivatives Market." *Manual for SOA Exam FM*. 2009. Web. Dec. 2016. <https://www.math.binghamton.edu/arcones/exam-fm/sect-7-1.pdf+>.

short position when Disney stock is selling for \$100. What is the profit of Ruthanne's investment?

Solution:

Ruthanne's initial proceeds from selling the borrowed stock are $20 * 106 = 2120$ and the total transaction cost is $20 * 2 = 40$. She purchased stock for $20 * 100 = 2000$ when the stock price dropped. Therefore, Ruthanne's total profit is $2120 - 2000 - 40 = \$80$.

c) Ask price, bid price, spread

The bid price is the price that the broker is willing to purchase the security and is therefore the selling price for an investor. The ask price is the price at which a broker is willing to sell a security. The ask price is the buying price paid by an investor. The difference in these two amounts is the bid-ask spread. This spread represents the broker's profit from the transaction. Typically the ask price is higher than the bid price so the broker will make a profit. The bid-ask spread can vary widely depending on the type of security and the market. The bid-ask spread can widen significantly in periods of market turmoil or during periods of illiquidity[23].

d) Maintenance margin, margin call

One way that investors can purchase stocks is by borrowing money and buying on margin. Typically the buyer borrows a portion of the price of the stock from a broker and contributes the remaining amount. The amount that the buyer must contribute is referred to as the initial margin requirement. Brokers frequently require a certain percentage contribution from the buyer when purchasing stocks on margin. The minimum amount of equity that must be maintained in a margin account is the maintenance margin. If the amount of equity dips below the maintenance margin, then the investor is at risk for a margin call. In this case, the investor must increase his or her equity in the account by contributing money to the account[24].

e) Put option

A put option is a financial instrument that grants the purchaser the right, but not the obligation, to sell a stock at a particular price for a specified amount of time. The price at which the put option can be sold at is known as the strike price. The length of time that the put option is valid for is the time to maturity, or expiration. There are two different types of put options: European and American. A European option can be exercised only at expiration whereas an American option can be exercised at any time up until maturity.

[23] Root. "Bid and Asked." *Investopedia.* 01 Apr. 2016. Web. Nov. 2016. <http://www.investopedia.com/terms/b/bid-and-asked.asp>
[24] Root. "Maintenance Margin." *Investopedia*. 05 Sept. 2015. Web. Jan. 2017. <http://www.investopedia.com/terms/m/maintenancemargin.asp>.

If the strike price is higher than the market price, then the put option should be exercised. However, if the strike price is lower than the market price, then the put option should not be exercised. Therefore, the payoff of a European put option is given by the equation:

$$(K - S(T))^+$$

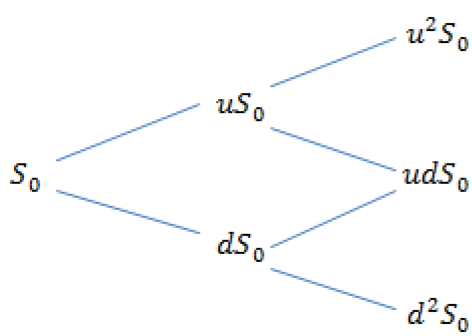
Where K is the strike price and $S(T)$ is the stock price at time T. To calculate the profit of investing in a put option, simply subtract the cost of purchasing the put option from the payoff.

To price a European put option, we can use either the put-call parity, or set up a replicating portfolio and use the binomial pricing model. If we know the price of a call on the same stock with the same strike price and expiration, we should use put-call parity as it is more efficient than the binomial pricing model. The put-call parity relates the price of the European call option, European put option, and the future value of the strike price if the money were invested in an interest bearing account. The formula for put-call parity is:

$$C - P = S_0 - \frac{K}{(1+r)^T} \quad \text{or} \quad C - P = S_0 - Ke^{-rT} \text{ if interest is compounded continuously,}$$

Where C is the price of the call option, P is the price of the put option, S_0 is the initial stock price, K is the strike price, r is the interest rate, and T is the maturity.

If we do not have the price of the call option, we will need to use the binomial pricing model to price the put option. The binomial pricing model makes the unrealistic assumption that the price of the underlying stock will either increase by an up factor or decrease by a down factor. In practice, there are infinitely many options for which the stock price to change, but this simplified model is still used to approximate the fair price of a put option. First, we must construct a "tree" that follows the behavior of the underlying stock. If we are given the initial stock price S_0, the up factor u, and the down factor d, then the stock "tree" is as follows:



The pattern can continue for more time periods, but for simplicity, we will stick to two time periods, or a maturity of $T = 2$. Next, we must calculate the probability of the stock price increasing by the up factor or decreasing by the down factor. Given the interest rate r along with the up and down factors, the probability of the stock price increase is given by the equation:

$$p = \frac{(1+r)-d}{u-d}$$

And to calculate the probability of the stock price decreasing, simply substitute the resulting probability into the equation $1-p$.

Now that we have the probabilities and resulting stock prices, we will use the payoff formula for a put option and work our way backwards to construct the "tree" for our put option. First, we must calculate the payoff of the put option at maturity for the various outcomes of the stock price. Then, we calculate the expected value of the payoffs using the two probabilities we calculated earlier. The put option "tree" is as follows:

$$(K-(u^2S_0))^+ = a$$

$$\frac{pa+(1-p)b}{i} = d$$

$$\frac{pd+(1-p)e}{i} \qquad\qquad (K-(udS_0))^+ = b$$

$$\frac{pb+(1-p)c}{i} = e$$

$$(K-(d^2S_0))^+ = c$$

Example 6.2:

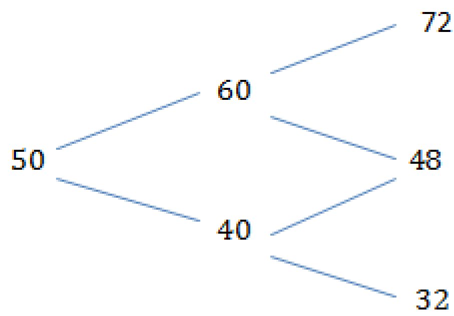
Meghan purchased a put option with a strike price of \$52 and a maturity of two months. The stock is currently trading for \$50. If the stock prices increases to \$54 in two months, what is the payoff of the put option? What is the fair price for this put option if the up factor is 1.2, the down factor is 0.8, and the interest rate is 2%?

Solution:

The payoff of the put option if the stock price increases to \$54 is:

$$(K-S(T))^+ = (52-54)^+ = 0$$

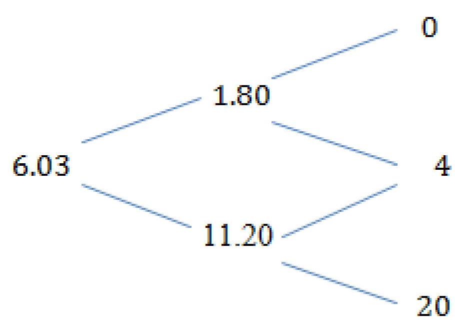
To calculate the fair price for this put option, we must first construct the stock price "tree"



Next, we calculate the probabilities, $p = \frac{(1+r)-d}{u-d} = \frac{1.02-0.8}{1.2-0.8} = 0.55$ and

$(1-p) = (1-0.55) = 0.45$

Finally, we will construct the put option "tree"



Therefore, the fair price for this put option according to the binomial pricing model is $6.03.

f) Call option

A call option is a financial instrument that grants the purchaser the right, but not the obligation, to buy a stock at a particular price for a specified amount of time. Similar to a put option, a call option has a strike price and expiration and there are both American and European call options that follow the same regulations as put options.

If the strike price is higher than the market price, the call option should not be exercised. However, if the strike price is below the market price, the call option should be exercised. Therefore, the payoff of a call option is given by the equation:

$$(S(T) - K)^+$$

Where K is the strike price and $S(T)$ is the stock price at time T. Similar to a put option, to calculate the profit of purchasing a call option, simply subtract the price of the option from the payoff.

Identical to a put option, we can use either put-call parity or the binomial pricing model to calculate the fair price of a call option. The only difference in using the binomial pricing model is the initial step of constructing the option "tree". Instead of using the formula for the payoff of a put option, we will use the formula for the payoff of a call option as stated above.

Example 6.3:

What is the payoff of a call option with strike price $52 if the stock price increases to $54 at maturity two months? What is the fair price of the call option if the up factor is 1.2, the down factor is 0.8, the interest rate is 2% compounded continuously, and the current stock price is $50?

Solution:

If the stock price increases to $54, the payoff of the call is $(S(T) - K)^{+} = (54 - 52)^{+} = 2$.

Since we already know that the put option with the same strike price and maturity trades at $6.03, we can use put-call parity to calculate the fair price of the call option.

$$C - P = S_0 - Ke^{-rT}$$

$$C - 6.03 = 50 - 52e^{-0.2(2)}$$

$$C = \$21.17$$

g) Moneyness

A financial instrument is said to be either in the money, out of the money, or at the money depending on the profitability of the investment. If the investment is profitable, it is in the money. While an investment that loses value is said to be out of the money and an investment that does not have any loss or gain is at the money. A call option is in the money if the stock price is higher than the strike price while a put option is in the money if the stock price is lower than the strike price. Therefore, when purchasing call option, the investor is bullish about the position and believes that the stock price will increase whereas the purchaser of a put option is bearish and is betting that the stock price will decrease.

Section 7: Forward Contracts, Futures, and Swaps

Forward contracts, futures, and swaps are other types of derivatives that are commonly used for hedging and speculation. There are several ways in which buyers and sellers of these derivatives can profit from the transactions, and investing in such derivatives can enhance a portfolio greatly.

a) Forward contract

A forward contract is a customized agreement between two parties to buy or sell an asset at a specified price on a future date. Since this contract is customizable, it is particularly useful when hedging. The contract can be constructed for a specific commodity, amount, and delivery date. The price of the asset in the forward contract is referred to as the forward price. Similar to options and other types of derivatives, there is typically an asset on which the value of the forward contract is based, and is known as the underlying asset. The delivery date is known as the expiration date. For example, a forward contract of 100 bushels of wheat to be delivered three years from now has an underlying asset of wheat, quantity of 100 bushels, and an expiration of three years.

A traditional forward contract requires no money up front, other than commission, and the money is delivered at expiration. The current price of the asset when the contract is entered is known as the spot price. The difference between the spot price and the forward price is the forward premium or forward discount. If the forward price is higher than the spot price, there exists a forward premium. On the other hand, if the forward price is lower than the spot price, the asset is forwarded at a discount. The buyer of the asset in a forward contract is called the long forward while the seller of the asset is the short forward. The profits of the long and the short in a forward contract are opposite of each other. Therefore, a forward contract is a zero-sum game[25].

Forward contracts are not traded on a centralized exchange and are therefore referred to as over-the-counter financial instruments. The ease at which these contracts can be created due to the over-the-counter nature increases the convenience of this financial instrument, but also increases the default risk of the investment. As such, forward contracts are not readily available to retail investors as other financial instruments[26].

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[25] "Derivatives Market." *Manual for SOA Exam FM*. 2009. Web. Dec. 2016. <https://www.math.binghamton.edu/arcones/exam-fm/sect-7-1.pdf+>.

[26] Picardo, CFA Elvis. "Forward Contract." *Investopedia*. 03 Apr. 2015. Web. Jan. 2017. <http://www.investopedia.com/terms/f/forwardcontract.asp>.

Example 7.1:

Bob enters into a forward contract with Fred to purchase 100 ounces of silver in one year at \$18 per ounce. What are Bob and Fred's payoffs if the spot price of the silver is \$15, \$17, \$20?

Solution:

Bob's payoff chart:

Spot Price	15	17	20
Payoff	100(-2) = -200	100(-1) = -100	100(2) = 200

Fred's payoff chart:

Spot Price	15	17	20
Payoff	100(2) = 200	100(1) = 100	100(-2) = -200

b) *Futures contract*

A futures contract is a legal agreement made on a trading floor of a futures exchange, to buy or sell a commodity at a predetermined price at a specific time in the future. Unlike forward contracts, futures contracts are standardized, and thus can be traded on a futures exchange. The largest future exchanges are the Chicago Mercantile Exchange, the Chicago Board of Trades, the International Petroleum Exchange of London, and the New York Mercantile Exchange[27].

In addition to outlining the quantity of the commodity being traded, a futures contract can also include a description of the quality of the commodity. The terms futures and futures contract are typically used interchangeably, and mean virtually the same thing.

Futures contracts are used either for hedging or speculating purposes. Producers and purchasers of an underlying asset hedge or guarantee the future price of the commodity. On the other hand, portfolio managers or traders tend to bet on the price movements of the underlying assets. There are two risks associated with a futures contract which include market risk and credit risk. The market risk refers to the volatility of the price of the asset, while the credit risk is related to the solvency of the partners within the contract. Market risk is a non-diversifiable risk, meaning that it cannot be eliminated. However, credit risk can be minimized by requiring a deposit to an account known as a margin account. The deposit amount is the initial margin which is determined by the exchange. If there is a loss on the

[27] Root. "Futures." *Investopedia*. 23 Nov. 2003. Web. Jan. 2017. <http://www.investopedia.com/terms/f/futures.asp>.

future, the investor's broker transfers that amount from the investor's margin account to the clearinghouse. If there is a profit, the clearinghouse transfers the amount to the investor's margin account. The value of the investor's margin account after settlement is given by the formula:

$$M_{t-\frac{1}{365}} e^{\frac{r}{365}} + N\left(S_t - S_{t-\frac{1}{365}}\right)$$

Where $M_{t-\frac{1}{365}}$ is the previous day's balance of the margin account, *r* is the interest rate, *N* is the nominal amount, S_t is the current price, and $S_{t-\frac{1}{365}}$ is the previous day's price.

Unlike forward contracts which are typically made on commodities, futures contracts can be made on a number of underlying assets including stocks market indices, currency pairs, interest rates, and just about every commodity produced[28].

Example 7.2:

On January 1, 2017, Molly entered a long futures contract for 200 barrels of crude oil priced at \$55. The margin account is 35% of the market value of the futures' underlying asset. The annual continuously compounded interest rate is 4%. On January 2, 2017, the price decreases to \$54 per barrel. What is the balance of Molly's margin account after the settlement?

Solution:

The initial balance in Molly's margin account is:

$$(200)(0.35)(55) = 3850$$

On January 2, the value of Molly's margin account is:

$$M_{t-\frac{1}{365}} e^{\frac{r}{365}} + N\left(S_t - S_{t-\frac{1}{365}}\right)$$

$$3850e^{\frac{0.04}{365}} + 200(54 - 55) = \$3650.42$$

c) Swap

A swap is a derivative contract in which two similar financial instruments that behave differently are exchanged. The two instruments being exchanged are called the legs of the swap. Two common types of swaps include commodities, and interest rate swap consisting of

[28] "Derivatives Markets." *Manual for SOA Exam FM*. Web. Jan. 2017. <https://www.math.binghamton.edu/arcones/exam-fm/sect-7-3.pdf+>.

a fixed rate in exchange for an adjustable or variable rate[29]. Typically, a swap includes a quantity that is known in advance, referred to as the fixed leg, and a quantity that is uncertain or unknown in advance, known as the floating leg. A swap will often times include an exchange of payments over time[30].

Similar to forward contracts, swaps do not trade on exchanges, but occur between business and financial institutions. Swaps are traded over-the-counter, and are not typically used by retail investors.

Commodity swaps involve the exchange of a floating commodity price for a set price over an agreed-upon term. Commodity swaps frequently include crude oil since oil prices are highly volatile; this type of swap is typically used to hedge against the price of a commodity. The floating-leg component of the swap is tied to the market price of the underlying commodity. The fixed-leg component is dependent on the contract agreement. In general, the floating-leg component is held by the consumer of the commodity willing to pay the fixed-price, and the fixed-leg is held by the producer who agrees to pay a floating rate. Therefore, the consumer gets a guaranteed price and the producer is protected from a decline in the price of the commodity[31].

Example 7.3:

ABC Corporation needs to purchase 1,000 barrels of crude oil per year for the next three years. The forward prices for delivery on oil in one year, two years, and three years are $49, $50, and $51 per barrel respectively. The one year, two year, and three-year spot rates are 1.5%, 2%, and 2.5%. Calculate the entire cost upfront of the oil and the cost of paying each year upon delivery.

Solution:

The upfront cost per barrel is:

$$\frac{49}{(1+0.015)}+\frac{50}{(1+0.02)^2}+\frac{51}{(1+0.025)^3}=\$143.69$$

Thus, by paying $\$143.69(1000)=\$143{,}692.87$ today, ABC Corporation is guaranteed 1,000 barrels of crude oil per year for three years. However, there is the risk that the oil will not be delivered so ABC Corporation may decide to make three payments, one each year as

[29] Root. "Swap." *Investopedia*. 30 Dec. 2015. Web. Jan. 2017. <http://www.investopedia.com/terms/s/swap.asp>.

[30] "Derivatives Markets." *Manual for SOA Exam FM*. Web. Jan. 2017. <https://www.math.binghamton.edu/arcones/exam-fm/sect-7-10.pdf+>.

[31] Root. "Commodity Swap." *Investopedia*. 19 Nov. 2003. Web. Jan. 2017. <http://www.investopedia.com/terms/c/commodityswap.asp>.

the barrels are delivered. The yearly cost per barrel X is calculated with the following equation:

$$143.69 = \frac{X}{(1+0.015)} + \frac{X}{(1+0.02)^2} + \frac{X}{(1+0.025)^3}$$

Therefore, by using computer software to solve for X, $X = \$49.58$ is the barrel cost that ABC Corporation must pay each year.

Interest rate swaps are slightly more complex than commodity swaps, as interest rates can be directly impacted by the Federal Reserve[32]. In an interest rate swap, the parties exchange cash flows based on a theoretical principal amount in order to hedge against risk or speculate changes in interest rates. The London Interbank Office Rate (LIBOR) is frequently used in interest rate swap agreements. This rate is published daily by the British Bankers Association and is based on rates that large international banks in London offer each other. The interest rate used in an interest rate swap is the most recent value of the LIBOR plus a margin[33].

Example 7.4:

ABC Company is willing to pay XYZ Incorporated an annual rate of LIBOR plus 1.4% on a theoretical principal of $750,000 for four years. In exchange XYZ pays ABC a fixed annual rate of 5% on the theoretical principal of $750,000 for four years. The LIBOR rate is currently 1.7%. Determine what happens if the LIBOR rates increase by 0.5% per year and 2.5% per year.

Note: XYZ Inc. benefits from the swap if rates rise significantly over the next four years and ABC Company benefits if rates fall, stay stagnant, or rise gradually.

Solution:

If LIBOR rises by 0.5% per year XYZ Inc.'s total interest payments to its bond holders over the four-year period are:

$$750000(4 * 0.014 + 0.017 + 0.022 + 0.027 + 0.032) = \$115{,}500$$

If the LIBOR rate had remained flat, the total interest payments would have been:

$$750000(4 * 0.014 + 0.017) = \$54{,}750$$

Therefore, XYZ must pay out an additional $115{,}500 - 54{,}750 = \$60{,}750$.

[32] "Derivatives Markets." *Manual for SOA Exam FM*. Web. Jan. 2017. <https://www.math.binghamton.edu/arcones/exam-fm/sect-7-10.pdf+>.
[33] Root. "Swap." *Investopedia*. 30 Dec. 2015. Web. Jan. 2017. <http://www.investopedia.com/terms/s/swap.asp>.

XYZ Inc. pays ABC Company $750000 * 4 * 0.05 = \$150{,}000$ and receives $\$115{,}500$ in return. Therefore, XYZ's net loss on the swap is $34,500.

If LIBOR instead rises by 2.5% per year, XYZ Inc.'s total interest payments to its bond holders over the four-year period are:

$$750000(4 * 0.014 + 0.017 + 0.042 + 0.067 + 0.092) = \$205{,}500$$

ABC pays this amount to XYZ and XYZ pays $150,000 in return. Thus, XYZ's net gain on the swap is $55,500.

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