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Complementary Uses of the Magnetic Vector Potential for the Understanding and Teaching of Electromagnetics

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Abstract

This work reports on uses of the magnetic vector potential $\mathbf{A}$ for understanding, visualizing and teaching electromagnetic phenomena. The teaching benefits of this work include examples of numerical modeling that facilitate understanding, and complemenatary visualization approaches. To facilitate discussion of $\mathbf{A}$ we first include a brief section looking at ϕ in a system comprised of two hollow, positively charged spheres, demonstrating that the work required to bring the two spheres close together $Q\phi$ is equal to change in the total integrated electric field energy density. We then introduce a system comprised of a hollow, positively charged sphere interacting with two parallel, current carrying wires, for which $\mathbf{A}_{\text{wires}}$ is determined as a function of position. Two calculations of the field momentum of the system are compared and shown to be equal: the first is $Q\mathbf{A}_{\text{wires}}$, the second is the integration over all space of the interaction momentum density $\epsilon_0\mathbf{E} \times \mathbf{B}$. We conclude by analyzing the trajectory of Q near the wires, establishing a visual connection between particle motion and the magnetic vector potentials sourced by the wires, including a discussion of relativistic implications.

1 Introduction

The study, calculation, and visualization of electromagnetic fields are of fundamental importance practically, for basic research, and in teaching. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are generally considered the primary fields, and knowing these we can calculate the forces on moving charges via: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$. Another tool to improve understanding of electromagnetic fields is the use of the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$. These can be related to $\mathbf{E}$ and $\mathbf{B}$ via the relations: $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

While most workers agree on the validity of ϕ as a standalone field quantity, some find the usefulness of $\mathbf{A}$ to be limited to being a mathematical construct used to find the "real" field $\mathbf{B}$. Others point out that there are phenomena such as the Aharonov-Bohm effect [1] that can be understood classically using a line integral of $\mathbf{A}$ in a region where $\mathbf{A}$ exists and $\mathbf{B}$ is zero. A significant number of workers emphasize the fundamental importance and practical usefulness of the potentials [2, 3, 4, 5, 6, 8, 7, 9, 10]. Emil Konopinski in particular was very passionate on this topic [5, 6], and his work inspired much of the work reported in this paper. That said, in electromagnetics textbooks $\mathbf{A}$ is usually considered as a calculation tool – Semon and Taylor provide interesting historical context for this point [11].

The intention of the work reported in this article is to expand on earlier work demonstrating that $\mathbf{A}$ can serve as a very useful, complementary lens for the understanding of electromagnetics, aided in particular by modern computational methods. In Section II we examine a basic, undergraduate-level system comprised of two hollow, positively charged spheres, and compare two approaches for determining the stored electric field energy in the system - we include this section as it lays the groundwork for the following sections. In Section III we introduce a two-wire system and calculate the field momentum resulting from the interaction between a hollow, charged sphere Q and the wires. In Section IV we consider a dynamic case where Q is in motion with respect to the wires. We conclude with a discussion of relativistic considerations.

2 Equivalence of $Q\phi$ and stored electric field energy

In this section we compare two approaches for determining the stored electric field energy due to the interaction of two hollow, positively charged spheres Q_1 and Q_2, with Q_1 located at the origin. The first approach to evaluating the stored electric field energy is to determine the work needed to bring Q_2 to a position $\mathbf{r}$ relative to Q_1 – this is an undergraduate level calculation. We compare this standard result with the less common approach of integrating the interaction electric field energy due to the two spheres over all space.

The potential as a function of position due to a hollow sphere of charge Q_1 centered at the origin is given by $\phi_1(\mathbf{r}) = Q_1/(4\pi\epsilon_0|\mathbf{r}|)$, where $\mathbf{r}$ is outside of the sphere. The work needed to bring Q_2 from infinity to a position $\mathbf{r}$ is W $= Q_2\phi_1(\mathbf{r}) = Q_2Q_1/(4\pi\epsilon_0|\mathbf{r}|)$. The work added to the system is considered to be potential energy stored as additional electric field energy [2, 12]. We will utilize Cartesian coordinates throughout, and so $\mathbf{r} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z$. As a concrete example we chose Q_1 and Q_2 as 1.0 x 10^{-12} C, with radii of 1.0 mm. In this work we use Mathematica to perform the numerical modeling and generate the plots.

The second calculation approach is to first find the total electric field energy density, $\mathrm{w}_{\mathrm{total}}(\mathbf{r})$, as a function of position. We find $\mathrm{w}_{\mathrm{total}}(\mathbf{r})$ as $\frac{1}{2}\epsilon_0\mathbf{E}_{\mathrm{total}}(\mathbf{r}) \cdot \mathbf{E}_{\mathrm{total}}(\mathbf{r})$, where $\mathbf{E}_{\mathrm{total}}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})$:

$$\mathrm{w}_{\mathrm{total}}(\mathbf{r}) = \frac{1}{2}\epsilon_0[\mathbf{E}_1(\mathbf{r})^2 + 2\mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})^2] \tag{1}$$

The first and third terms are the energy densities due to Q_1 and Q_2, respectively, and the cross term in the middle is the energy density due to the interaction of the two spheres, which we denote as $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$.

In figure 1, $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$ is shown for the case where Q_2 is located at $\hat{\mathbf{i}}4.0$ mm. Note that the interaction energy is negative midway between the spheres because $\mathbf{E}_{\mathrm{total}}$ is zero there. Above Q_2 or below Q_1, $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$ add, and so $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$ is positive there.

To find the total interaction energy $W(x_0)$ associated with Q_2 brought from infinity to a given location x_0 on the x-axis, we integrate the interaction

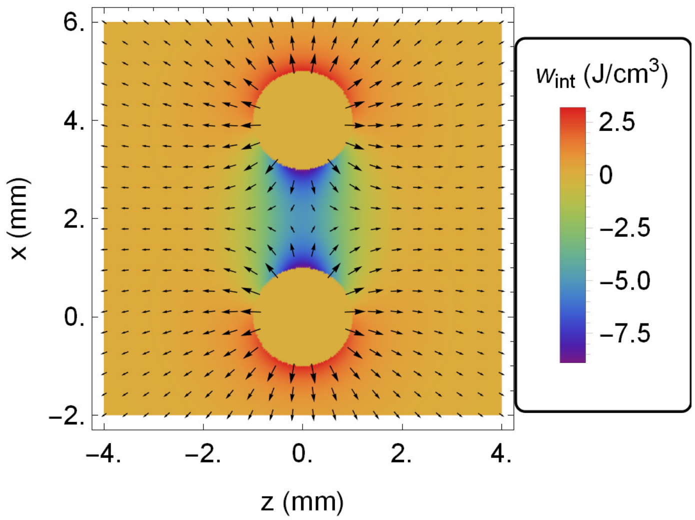

Figure 1: Interaction energy density, $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$, near Q_1 and Q_2. Also shown are the $\mathbf{E}_{\mathrm{total}}(\mathbf{r})$ vectors, except inside the spheres. As the interior of spheres are hollow, $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$ evaluates to zero there.

energy density over all space:

$$W(x_0) = \int_V \mathrm{w}_{\mathrm{int}}(\mathbf{r}) dV = \epsilon_0 \int_V \mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r})\ dV \tag{2}$$

We have carried out calculations for $x_0 = 4$ to 20 mm for both methods. We find the calculated results to be in excellent agreement as they must be – for each value of x_0 the two approaches differ by no more than $1.0\,\mathrm{x}10^{-6}$ %.

The equivalence of $Q_2\phi_1(x)$ and $\int_V \mathrm{w}_{\mathrm{int}}(\mathbf{r})\, dV$ provides a complementary lens for understanding the interaction between a charge and a preexisting electric potential. Although the units of ϕ are usually given as V, an equivalent unit is J/C. $Q\phi$ is then J, and this point quantity is a surrogate for the integrated $\mathrm{w}_{\mathrm{int}}(\mathbf{r})$. So while $\mathbf{E}(\mathbf{r})$, for which the dimension can be given as N/C, represents the force per unit charge at $\mathbf{r}$, $\phi(\mathbf{r})$ in J/C represents the stored electric field energy per unit charge at $\mathbf{r}$. A plot of the interaction energy as a function of position as in figure 1 provides another way for a student to get a feeling for the underlying physics. Although still abstract, integrating the interaction energy density as a function of position is more "visualizable" than simply stating that the potential energy is $Q\phi(\mathbf{r})$. We express the equivalency of the two approaches as:

$$Q\phi(\mathbf{r}) = \int_{\text{all space}} \mathrm{w}_{\text{int}}(\mathbf{r}')\, dV' \tag{3}$$

3 Equivalence of $Q\mathbf{A}$ and stored electromagnetic field momentum

In this section we compare the point quantity $Q\mathbf{A}(\mathbf{r})$ which has units of momentum with the results of integrating the corresponding momentum density of the fields. The model system utilized is comprised of two parallel, hollow, current carrying wires and a hollow, positively charged sphere Q, where we consider the interaction between Q and $\mathbf{A}(\mathbf{r})$ due to the wires. As a concrete example we choose z-directed wires located in the xz plane, passing through x_{wire} = +/- 5.0 mm, with radii 0.5 mm, and with I = +/-100.0 A. An advantage of this two-wire system is that it allows a natural choice for $\mathbf{A}(\mathbf{r}) = 0$, which is the z-axis. In our simplified teaching model we assume that the voltage is equal along and between the wires, and also that Q does not induce significant image charges, so that there is no net charge on the wires. In a real system there *are* net charges on the wires, although modeling these is beyond the scope of this paper, and we believe that keeping the model simple facilitates teaching. The geometry used in this and the following section is shown in figure 2.

The sphere Q is stationary at $\mathbf{r}_0 = \hat{\mathbf{i}}x_0$, and discrete calculations are made over a range of values of x_0. We use two methods to evaluate the interaction between Q and the vector potential due to the wires: the first method is to find $\mathbf{A}(\mathbf{r})$ directly, from which the associated field momentum is calculated as $\mathbf{P} = Q\mathbf{A}(\mathbf{r})$ as shown in [6]. The second method is to integrate the field momentum density $\epsilon_0(\mathbf{E} \times \mathbf{B})$ throughout the volume of interaction.

To find $\mathbf{P} = Q\mathbf{A}(\mathbf{r})$ we first find $\mathbf{B}(\mathbf{r})$, and then determine $\mathbf{A}(\mathbf{r})$. First we note that $\mathbf{A}(\mathbf{r})$ must be z-directed, as can be seen from equation 4 below which is valid in the nonrelativistic regime under the Coulomb gauge, $\nabla\cdot\mathbf{A} = 0$:

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 c^2} \int_{V'} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV'. \tag{4}$$

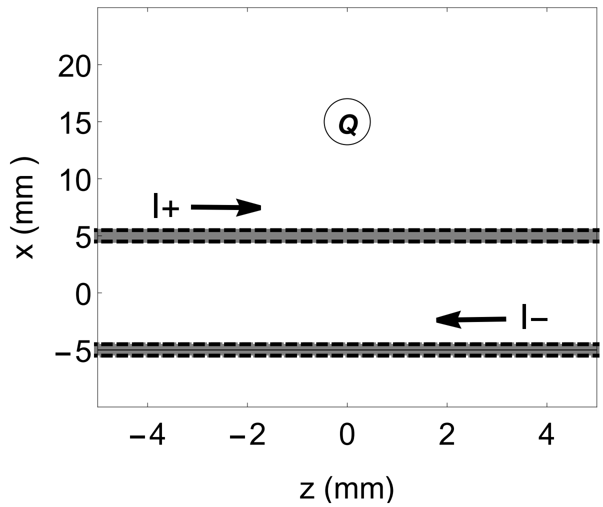

Figure 2: Shown is the geometry used for Sections III and IV. The currents on the two wires are equal in magnitude and opposite in direction.

In our model system the wires are parallel to the z-axis, so that $\mathbf{j}$ is z-directed and in turn $\mathbf{A}$ must have the form $\mathbf{A}(\mathbf{r}) = \hat{\mathbf{k}}A_z(\mathbf{r})$. We note that in this two-wire system with equal and opposite currents on the two wires the magnitude of $\mathbf{A} \to 0$ as $r \to \infty$. Due to symmetry, $\mathbf{A} = 0$ on the z-axis.

With charge Q located on the x-axis in this first method for determining $\mathbf{P}$ we only need $\mathbf{B}(\mathbf{r})$ in the xz plane. Given the assumption that the wires are effectively infinite in the z-direction, in the xz plane $\mathbf{B}(\mathbf{r})$ will depend only on x and will have a y-component only, so that $\mathbf{B}(\mathbf{r}) = \hat{\mathbf{j}}B_y(x)$. Using Ampere's law, the combined magnetic field from the two wires in our model system becomes:

$$B_y(x) = \frac{I}{2\pi\epsilon_0 c^2(x - x_{\text{wire}})} - \frac{I}{2\pi\epsilon_0 c^2(x + x_{\text{wire}})}. \tag{5}$$

Because $\mathbf{B}$ only has a y-component, and $\mathbf{A}$ only has a z-component, $\mathbf{B} = \nabla \times \mathbf{A}$ reduces to:

$$\hat{\mathbf{j}}B_y(x) = -\hat{\mathbf{j}}\frac{\partial A_z}{\partial x}. \tag{6}$$

Using equation 6, we can now integrate from $x = 0$ out to the desired value of x_0 to determine $A_z(x_0)$. Because $A_z = 0$ on the z-axis the constant of integration in equation 7 will evaluate to zero. We proceed as follows:

$$A_z(x_0) = \int_0^{x_0} \frac{\partial A_z(x')}{\partial x'}\, dx' = \int_0^{x_0} -B_y(x')\, dx'. \tag{7}$$

Knowing $B_y(x')$ from equation 5, we can readily find $A_z(x_0)$ using equation 7. This approach to finding $A_z(x_0)$ is analogous to integrating $\mathbf{E}(\mathbf{r})$ to find $\phi(\mathbf{r})$. Since here $\mathbf{A}(\mathbf{r})$ has only a z-component, with Q located at $\mathbf{r}_0 = \hat{\mathbf{i}}x_0$ the total field momentum becomes $\mathbf{P} = \hat{\mathbf{k}}P_z = \hat{\mathbf{k}}QA_z(x_0)$. We present the results for $\mathbf{P}$ below, along with the results calculated via a volume integration.

The second method for determining $\mathbf{P}$ uses the momentum density of the electromagnetic fields. The Poynting vector $\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$ is familiar as the power density – the rate at which energy is transported per unit surface, with units $Js^{-1}m^{-2}$. Following Griffiths, (see [12], 8.2.2), scaling $\mathbf{S}$ by $1/c^2$ yields the field momentum density $\epsilon_0(\mathbf{E} \times \mathbf{B})$, expressed as $\mathbf{g}$. The total field momentum stored in a volume is found as $\mathbf{P} = \int_V \mathbf{g}\, dV$. In our system Q is stationary and so does not have a magnetic field, and as the wires do not have an electric field then $\mathbf{g}$ is *only* due to the interaction between wires and Q. In figure 3, $\mathbf{g}$ is plotted for the case where the closer edge of the sphere is 0.5 mm above the edge of the upper wire. Calculated $\mathbf{g}$ is highest directly between the wire and the sphere, and is in the negative z-direction there. When we integrate $\mathbf{g}$ over all space, however, the net result for $\mathbf{P}$ is positive z-directed for sphere locations Q above the wire.

The calculation results for $Q\mathbf{A}(\mathbf{r})$ and $\mathbf{P} = \int_V \mathbf{g}\, dV$ are shown in figure 4. The agreement is excellent as it must be – here they are within 0.004% or better. This agreement parallels the equivalence of $Q\phi(\mathbf{r})$ to $\int_V \mathrm{w_{int}}(\mathbf{r})\, dV$ as discussed in Section II. This equivalency can be expressed as:

$$Q\mathbf{A}(\mathbf{r}) = \int_{\text{all space}} \mathbf{g}\, dV'. \tag{8}$$

Although we are considering a series of stationary locations for Q, the sphere will experience experience a -z-directed magnetic force as it is moved from one location to the next. To move the sphere perpendicularly towards the wires positive z-directed field momentum must be added to the system, analogous to needing to add stored electric field energy in Section 2. Our example is similar to [7], except they consider stationary Q near a wire, and slowly

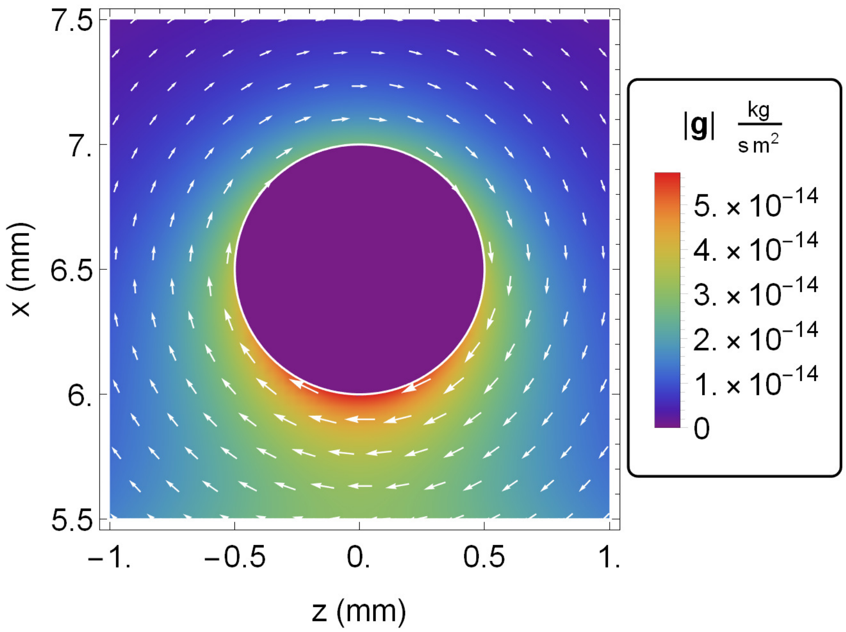

Figure 3: Plotted are the momentum density vectors $\mathbf{g}$ surrounding the sphere: their direction is represented by the arrows, their magnitude is represented by the color scale. $|\mathbf{g}|$ is greatest directly between the wire and the sphere, and zero inside the sphere since $\mathbf{E} = 0$ there. The top of the wire is at $x = 5.5$ mm.

increase the current from 0, needing to add field momentum to hold Q in place.

4 Particle Trajectory

In this section we consider the trajectory of a small sphere of charge Q near a two-wire system having the same geometry as in Section 3. Here we regard the system as closed, so that the total momentum of the system must be conserved throughout the trajectory. The mass of Q as well as its initial velocity and position are chosen such that the sphere remains near the wires. The wire masses are assumed infinite in comparison with m_Q. We discuss visualization and teaching opportunities related to this dynamic system and

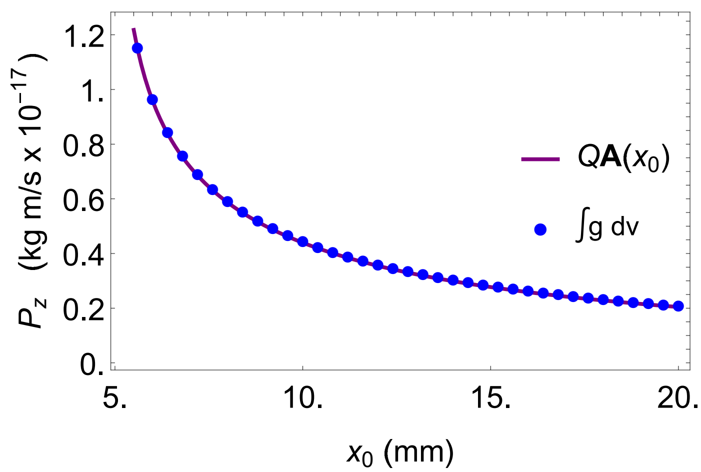

Figure 4: Comparison of two methods to calculate $\mathbf{P}$ associated with Q over a range of x_0 values. $Q\mathbf{A}(x_0)$ is $\mathbf{P}$ calculated directly using the value of $\mathbf{A}(x_0)$[5]. $\mathbf{P} = \int_V \mathbf{g}\ dV$ is the integrated field momentum density. Plotted are the results for P_z, the only nonzero component of $\mathbf{P}$.

conclude with relativistic considerations.

To facilite demonstration of the balance between field and particle momenta, we choose initial values of $\mathbf{v} = 20.0\hat{\mathbf{k}}$ m/s and $\mathbf{r} = 20.0\hat{\mathbf{i}}$ mm and also choose $m_Q = 5.0$ x 10^{-19} kg and $Q = 2.0$ x 10^{-12} C. Because $\mathbf{B}(\mathbf{r}) = \hat{\mathbf{j}}B_y(\mathbf{r})$ in the xz plane, $\mathbf{F}_{\text{magnetic}} = Q[\mathbf{v} \times \hat{\mathbf{j}}B_y(\mathbf{r})]$, and $\mathbf{F}_{\text{magnetic}}$ will always be perpendicular to $\hat{\mathbf{j}}$. Knowing that the y-component of the force is always zero in the xz plane, then given an initial velocity in the z-direction and an initial location in the xz plane, Q remains in the xz plane and $v_y = 0$ always. Therefore:

$$\mathbf{F}_{\text{magnetic}}(\mathbf{r}) = Q\mathbf{v} \times \hat{\mathbf{j}}B_y(\mathbf{r}) = \hat{\mathbf{k}}Qv_xB_y(\mathbf{r}) - \hat{\mathbf{i}}Qv_zB_y(\mathbf{r}). \tag{9}$$

The trajectory is simulated using a timestep approach. With $\mathbf{a} = \mathbf{F}_{\text{magnetic}}/m_Q$, we update $\mathbf{r}, \mathbf{v}$, and $\mathbf{a}$ after each timestep, Δt. Details of the simulation are included in publicly downloadable Mathematica notebooks [14]. The results for the trajectory are shown in figure 5. The vectors represent the instantaneous velocity at various positions. The circle at position (1) represents the starting point, at which $x = 20.0$ mm, $v_z = 20.0$ m/s and $v_x = 0$. Initially $\mathbf{F}_{\text{magnetic}}$ is in the -x direction, so Q is accelerated towards the wire. As v_x

becomes increasingly negative, $\mathbf{F}_{\text{magnetic}}$ becomes stronger in the -z direction, such that by the time the particle reaches position (2) $v_z = 0$ and the velocity is only in the -x direction. At closer positions to the wire, v_z becomes increasingly negative while v_x increases such that by position (3) $v_x = 0$. The particle is then repelled upward to the top of the trajectory, after which the pattern repeats. In this system the sphere spends less time near the wire, since the magnetic force is stronger there. For the sphere to be repelled at point (3) there must be -$\mathbf{v}_z$ there, which is why we see the "loop" at the bottom of the trajectory – we see this as well with different initial $+\mathbf{v}_z$.

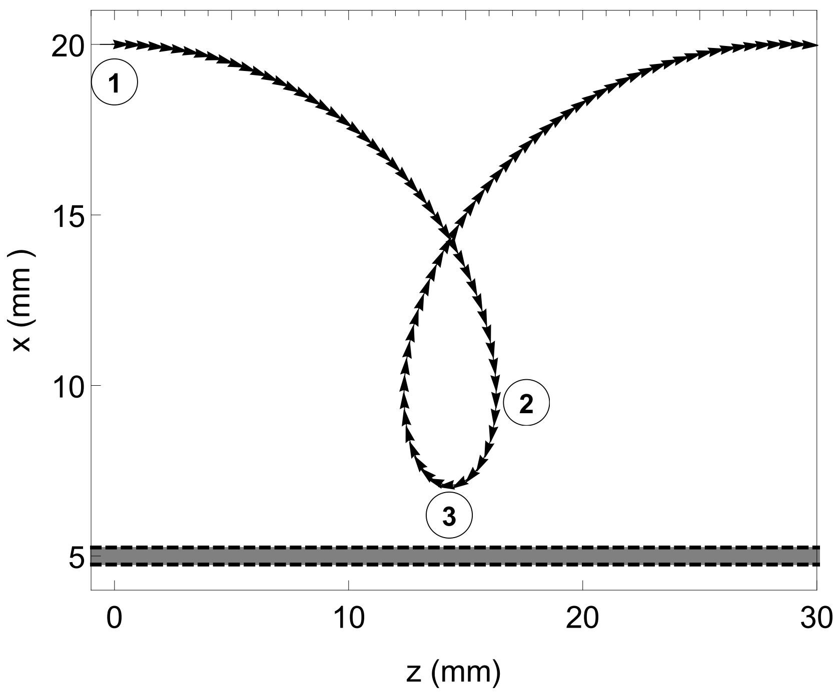

Figure 5: The particle trajectory is shown in the rest frame of the wires, which carry ± 100 A. The top wire is shown, and the wires exert a force $\mathbf{F}_{\text{magnetic}}$ on Q. Because $\mathbf{F}_{\text{electric}} = 0$ the kinetic energy is constant and $|\mathbf{v}_Q|$ is constant.

We now consider the field momentum $\mathbf{P}(\mathbf{r}, t)$ and the sphere's momentum $\mathbf{p}(\mathbf{r}, t) = m_Q\mathbf{v}$ during the course of the trajectory. In Section 3 we found the field momentum associated with a sphere of charge to be $\mathbf{P} = Q\mathbf{A}(\mathbf{r})$. In this section unlike in Section 3, we consider a closed system, and so the total system momentum, here including the wires, the sphere, and the field momentum, is constant. The "generalized momentum" includes only those quantities parts of the momentum directly associated with the particle - this is discussed in detail by Semon and Taylor [11], who go into detail for a Langrangian approach as well. The generalized momentum is expressed as [6, 11]:

$$\frac{d}{dt}(\mathbf{p}+\mathbf{P}) = 0. \tag{10}$$

In the present case, as in Section III, the currents that source $\mathbf{A}$ are z directed and so $\mathbf{A}(\mathbf{r}) = \hat{\mathbf{k}}A_z(\mathbf{r})$. We can write the z component of equation 10 as:

$$\frac{d(m_Q v_z + QA_z)}{dt} = 0 \tag{11}$$

From equation (11) we see that while the z-components of the particle and field momenta vary with time and position, their sum is constant. In this case we find the generalized momentum - those parts of the momentum directly associated with the particle – *is* conserved. Basically there is an ongoing exchange between $m_Q v_z$ and $P_z(\mathbf{r},t)$ throughout the trajectory - the fields act a a reservoir for z-directed momentum. Using our approach from Section III we determine $\mathbf{A}(\mathbf{r})$ which in this case is $\hat{\mathbf{k}}A_z(\mathbf{r},t)$, and from which the z-component of the field momentum is found as $P_z(\mathbf{r},t) = QA_z(\mathbf{r},t)$. Because $y = 0$ and there is no variation in the z-direction then $\mathbf{A}$ depends only on x. The z-component of the momentum of the sphere is $p_z = m_Q v_z(\mathbf{r},t)$. The z-components of the momenta of the fields and of the sphere, along with their sum, are shown in figure 6. Referring to figure 4 note that as the sphere moves from position (1) to position (3) in the trajectory (or from 20.0 to about 2.0 mm from the wire), the magnitude of QA_z remains positive and roughly triples. At the same time p_z decreases to the point where it switches direction and becomes negative, which is necessary for the total z-directed momentum to be conserved.

In the x-direction, however, the generalized momentum is *not* conserved, as there is no x-directed field momentum to balance changes in $m\mathbf{v}_x$. In this case, because the mass of the wires is assumed infinite in comparison to m_Q, the wires can absorb changes in p_x and remain effectively stationary, so that the total system momentum still conserved in the x-direction even though $P_x(\mathbf{r}, t) = 0$. By contrast, in the z-direction the momentum balance with p_z is fully accounted for by $P_z(\mathbf{r}, t)$, or to put it in another way the z-component of the generalized momentum is conserved.

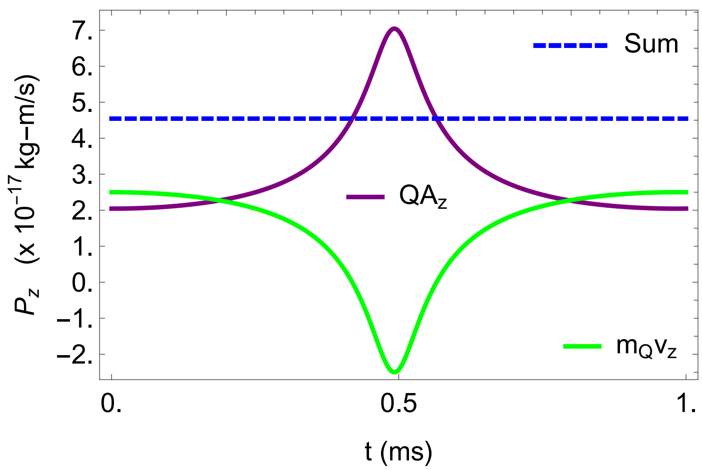

Figure 6: Shown are the z-component of the particle momentum, the field momentum, and their sum as functions of time. This sum is constant, in agreement with theory.

Turning to relativistic considerations, a classic example (see, for example [2, 13]) is to look at the magnetic force experienced by a charge Q traveling parallel to a current carrying wire, for example at locations (1) and (3) in figure 5. As the wires are charge neutral, there is no electric field in the rest frame of the wires. However, in the rest frame of Q there is a linear charge density on the wires that exerts a coulombic force on Q, while $\mathbf{F}_{\text{magnetic}} = 0$: often used as a demonstration of the underlying connection between $\mathbf{E}$ and $\mathbf{B}$.

We find that the relation $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) - \partial\mathbf{A}(\mathbf{r})/\partial t$ multiplied by Q so that it becomes an expression of electric force is helpful in this discussion:

$$\mathbf{F}_{\text{electric}} = Q\mathbf{E}(\mathbf{r}) = -\nabla[Q\phi(\mathbf{r})] - \frac{\partial[Q\mathbf{A}(\mathbf{r})]}{\partial t}. \tag{12}$$

At location (1), consider the rest frame of Q where the wires, now moving parallel to Q in the z-direction, are a constant distance from Q, and have a linear charge density. Because the distance between the wires and Q is not changing, $\mathbf{A}$ is constant in this case. Therefore only the first term on the RHS of Eq. (12) is nonzero, and the force is purely coulombic. Given that $Q\phi(\mathbf{r})$ is the total interaction energy W_{int}, and at (1) $\mathbf{F}_{\text{electric}} = -\nabla[Q\phi(\mathbf{r})]$ only, the force there is due to a gradient in W: $\mathbf{F}_{\text{electric}} = -\nabla W$. It is interesting to note that even when Q is travelling at less than $2.0 \times 10^{-7}c$, we must consider relativistic length transformations to understand the coulombic force on Q in its rest frame.

What about location (2) in figure 5 where Q is traveling directly towards the wire? In this case, in the rest frame of Q there is no length contraction resulting in a net linear charge density on the wires, however the wire is now traveling towards Q with $\mathbf{v} = +\hat{\mathbf{i}}v_x$ and so $\partial\mathbf{A}/\partial t \neq 0$. In the rest frame of Q the force on Q can only be electric, however this time Eq. (12) reduces to $\mathbf{F}_{\text{electric}} = -\partial[Q\mathbf{A}(\mathbf{r})]/\partial t$, and the force is purely inductive.

A complementary way to understand the force acting on the sphere at location (2) is consider the momentum balance between field and sphere. In Section 3 we found $Q\mathbf{A}(\mathbf{r})$ equals the system's field momentum, $\mathbf{P}$. Here $\mathbf{A} = \hat{\mathbf{k}}A_z$, so $\mathbf{F}_{\text{electric}} = -\hat{\mathbf{k}}\ \partial QA_z/\partial t = -\hat{\mathbf{k}}\ \partial P_z/\partial t$. Note that the sign of the force is correct, as it must be: at location (2) Q is traveling towards the wire with $\mathbf{v} = -\hat{\mathbf{i}}v_x$, and so QA_z and the associated field momentum P_z are increasingly positive. For the total z-directed momentum to be conserved (as shown in figure 6), at point (2) the z-component of the particle momentum must become increasingly negative, so the force on Q must be in the $-\hat{\mathbf{k}}$ direction. This agrees with the direction of $\mathbf{v} \times \mathbf{B}$ which is $-\hat{\mathbf{i}}v_x \times \hat{\mathbf{j}}B_y = -\hat{\mathbf{k}}v_xB_y$. We note that the electromagnetic interaction at point (1) differs from that at point (2), which may not be noticed when considering $\mathbf{E}$ from the perspective of the Lorentz force equation. In addition at (2) the the inductive force as discussed here may be easier for a student to understand than when using $\mathbf{B}$: instead of needing a curl to find $\mathbf{B}$ and then a cross product to find the force, $\mathbf{J}$ and $\mathbf{A}$ are in the same direction and then the force is $\hat{\mathbf{k}}\ \partial QA_z/\partial t$. The point that it can be helpful for student comprehension to avoid the curl is discussed at some length in [11].

5 Conclusions

In this paper we present results of our work looking at the use of $\mathbf{A}$ as complementary to the use of $\mathbf{E}$ and $\mathbf{B}$ in visualizing and understanding electromagnetics. In Section 2, in order to set the framework for the subsequent sections, we compare the work needed to bring two charged spheres together, $Q_2\phi$ – a standard undergraduate calculation – with the interaction energy integrated over the entire field, $W = \int_V \mathrm{w_{int}}\, dV$. We use an analagous approach in Section 3, this time first showing that momentum equal to $Q\mathbf{A}(\mathbf{r})$ must be added to the system when bringing a charged sphere towards two current carrying wires, and then comparing that result with the interaction momentum density integrated over all space, $\mathbf{P} = \int_V \mathbf{g}\, dV$. We find words from Emil Konopinski provide an appropriate summary of sections 2 and 3:

"*Whereas* $\mathbf{E}, \mathbf{B}$ *describe a field in terms of forces the field can exert on charged matter,* ϕ *and* $\mathbf{A}$ *describe the same field in terms of energies and momenta that the entire field makes available to the matter*" [6].

Section IV covers results for a closed system, with consideration of how evaluating momentum storage in the fields can help in visualizing the underlying physics. Section IV concludes with a discussion of relativistic considerations, emphasizing how the contribution of each term of $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) - \partial\mathbf{A}(\mathbf{r})/\partial t$ varies throughout the trajectory. Taken as a whole we believe our efforts affirm and extend points made by earlier workers.

From a teaching perspective, the calculations presented in this article are straightforward, and Mathematica notebooks to calculate and plot the results are available to freely use and modify [14]. The coding is fairly basic, and an undergraduate student with solid programming skills could develop their own similar models in any high level programming language or math package. The integrations and trajectory calculations in sections 3 and 4 would be difficult if not impossible without computer modeling, demonstrating the effectiveness of this approach to students. Finally, we believe that exposing students to different approaches to understanding the same physical situation can facilitate comprehension.

6 Acknowledgements

We honor Professor Emil Konopinski in memoriam, without whose efforts the current work would not have come to fruition [15]. We thank Professors Al Weatherwax, Chris Ruf and Mark Semon for valuable feedback during the course of this project, and the reviewers for their helpful comments on the manuscript.

7 Endnotes and references

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[14] Mathematica files are downloadable at: `https://drive.google.com/drive/folders/1xz7ZbBCBJ00dOnq_n4Ebc7uemIEyq4Oe?usp=share_link`.

[15] Biographical details of Professor Konopinski's life can be found at: `https://alliance.iu.edu/members/member/3485.html`.