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Behind the Tiles: Mathematics of Carcassonne

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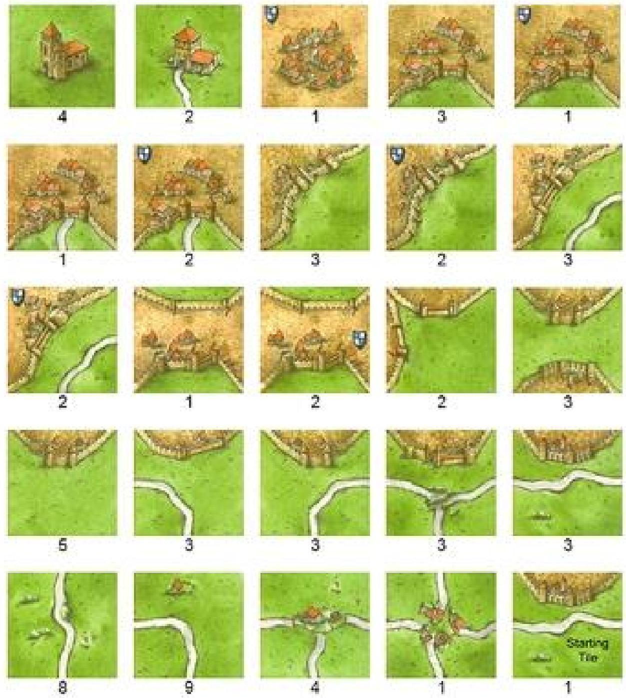
Abstract

Carcassonne is a tile-placing game where players take turns choosing a tile from a stack and attempting to create a city, road or a meadow. In addition to this, there is a river expansion pack that has river tiles to be placed. This paper focuses on how many different layouts or configurations of the river expansion pack can be created. It also discusses the Matlab code adapted to create a simulation of possible configurations of the river expansion pack.

Keywords: *Carcassonne, Matlab, Combinatorics, Edges, Vertices*

Carcassonne is a tile-based game for two to five players based on the French town of the same name. Each player draws a tile that features a part of landscape, like a road, city, or land, and with the expansion pack, a river. The goal of this game is to create a city, road, or farm by joining the different tiles together (Wredie, 1995). It is first important to understand the rules of the game before beginning layout calculations.

The rules are as follows: first, each person takes a turn and picks a tile from the stack. The stack is a combination of the possible tiles, faced down, so no player can predict what the next tile will be. When placing a tile on the board, it is imperative to attach one feature on the tile to another, i.e. road to a road, river to river, city to city, or meadow to meadow. Each player has a set of seven meeples, the figurines seen throughout the game. A player can choose to place a meeple on a tile while it is their turn if there is a road, city, farm, or a monastery that is not already claimed by either their own or another player's meeple. The meeple must stay on that location until it is completed, or the game runs out of tiles. Only then will the points be accredited to the player. A completed road is a road that ends either because it encountered a city, monastery, or feature that connects all the roads. A completed city is a city that has walls on all sides of it. A completed farm/meadow is a farm that is enclosed entirely by either roads or cities. A farm only counts for points if there is a city attached it, for the farm is the "source" of food for the city. Lastly, a monastery is only considered completed when it is surrounded on all sides.



Appelcline, S. (2006). Main Game.

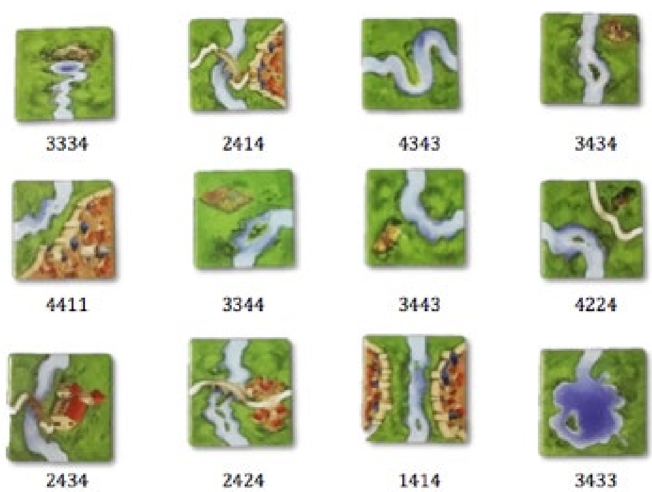
Each player has seven meeples to place. The points are given as follows: a city is one point per tile, two if there is a shield on the tile. A road is one point per tile, a monastery is one point per surrounding tile and can only be calculated when there are eight tiles surrounding the monastery. A farm is only calculated at the end of the game and points are awarded for each completed city the farm is attached to. For each city the farm is connected to, the farm is four points. Throughout the game, with only seven meeples, each decision is important.

When placing tiles for maximum point values, it is important to understand the features and categories of the tiles. Each tile is categorized by its edges, starting at the top edge. If each tile is placed down upright, the top edge or north edge, is first considered and recorded, and then each time moved clockwise and record the edge. That is how a straight river tile is differentiated from a curved river tile. Likewise, a tile with rivers and cities can be distinct from other tiles. The categorization of this process is influenced by *Carcassonne-Description of the Game* by Lucia Kárná (2012), mentioned later in this paper.

With the rules clarified, two articles were consulted for their research and calculations. One of the articles, *Carcassonne in the Classroom* by Mindy Capaldi and Tiffany Kolba (2017), categorized the different tile pieces by similarity, and then categorized them by giving each distinct tile a letter from the alphabet. It was important to know that while some tiles are mirror images of others, others were more distinct and thus categorized as two different types. It looked at seventy-one tiles, not including the starting tile which does not change from game to game.

Next the article looked at what is the minimum number of tiles where there is no valid placement, meaning a tile could not be placed anywhere where it would connect with other tiles but not interfere with the features. For example, a road cannot connect to a meadow. The minimum number was three tiles. Before, a feature of one tile did not connect with the feature of another tile. This was found by first classifying the tiles from the ABC list mentioned earlier, then observing how many of each type was found. When the authors ran a simulation, it was discovered that only one out of every fifty games ended with no valid placement for a tile (Capaldi & Kolba, 2017). This article looked at invalid placements, whereas the next article used a more combinatorics approach to look at valid placements.

The second article, *Carcassonne-Description of the Game,* by Lucia Kárná (2012), examines the features on the tiles, such as roads, cities, and meadows. This is something that is considered when writing code. It classified each tile by its edges. The article looked at the tiles using graph theory to interpret the edges and vertices of each tile when calculating possible configurations. This was mentioned earlier when categorizing tiles so that distinct and similar tiles are accounted for. Below is a figure of river tiles where "1" = city, "2" = road, "3" = meadow/farm, and "4" = river.



These articles presented different techniques for observing the tiles in Carcassonne. Both articles provided insight on how to begin characterizing the tiles. Although those articles did not mention or expand on how the math applies to the river expansion pack, it provides a foundation for

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expanding on the ideas of the two articles mentioned and arriving at new conclusions now including the river expansion pack. This paper will be observing how many different layouts of the river tiles can be created.

2 Classifying the Tiles: Stack Orders

There are twelve river pieces, one is the starting tile and the other the ending, for a total of ten tiles that have a choice of how they are picked from the stack of tiles during a game. Within those ten, there are six straight tiles and four curved tiles. Within these tiles, there are two straight tiles that are interchangeable, where if they were swapped for one another it would not change the layout of the game at all.

Likewise, for the curved tiles there are two curved tiles that can be swapped for one another that will not change the layout at all.

That may seem irrelevant, but when placing the tiles specifically, it comes into account with the different configurations. To look into this further, consider the word QUICKENING. It has ten letters, similar to the ten tiles that a player may choose from. If each letter is treated like a tile, they create a "stack" like in the game. In order to determine how many stack orders of quickening are possible, first regard the ten slots that the letters may choose from. These slots represent locations in the stack order:

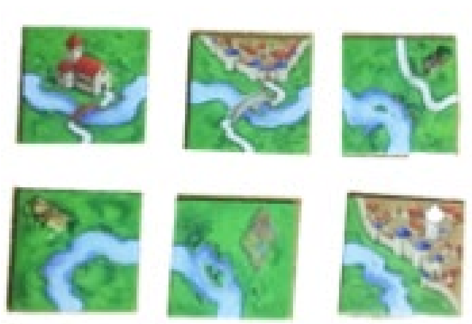
— — — — — — — — — —.

For the first slot, there are ten letters available to choose, then the second slot has nine because one letter is already picked, then the next one has eight slots, and so forth. This leads to $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$ stack orders. Therefore, the calculation represents a permutation of ten distinct objects; however not all the objects, or letters, are distinct.

Consider the I's and the N's. Those represent the two straight tiles and two curved tiles that can swap places for one another. Change the layout of QUICKENING: CUIQKNIENG. Note, the two Is are in two distinct places. However, if the Is were swapped CUIQKNIENG would become: CUIQKNIENG. The word does not change despite the Is being swapped. Likewise, if the Ns were swapped CUIQKNIENG would become CUIQKNIENG. The word again did not change even though the Ns were swapped. Therefore by doing the 10!, the Is were double counted when CUIQKNIENG was arranged twice even with the Is swapped. That means there are 2! ways to rearrange I among slots that contain I. The Ns were also double counted, with 2! ways to rearrange the N among slots that contain N. Therefore, $\frac{10!}{2!2!}$= 907,200 accounts for the total number of different stack orders.

3 Interpretations

The stack order affects how the game is played. For instance, if someone draws a curved tile and then another person draws a curved tile, this is a different layout than if a curved then a straight tile were drawn. Another thing to consider when calculating the number of possible layouts of the river expansion pack, after deciding the stack order, is the rotation of the tiles. There are six main tiles that need to be considered about the rotation of tiles. This leads to the six tiles that when rotated, affect the layout of the game, otherwise known as the six directed tiles. They are called directed tiles because the direction the tile is placed affects the configuration of the game.

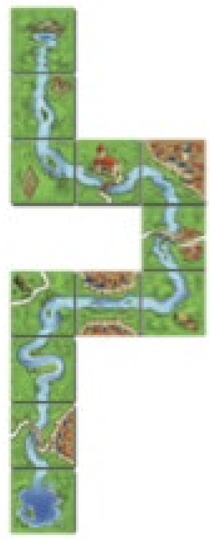


Directed Tiles

Non Directed

These six directed tiles comprise of four curved tiles and two straight tiles, which are tiles that when rotated will affect the flow of the river or the layout of the rest of the game. The other river tiles are, for the most part, symmetrical, therefore rotating them will not change how the game is played.

That may seem irrelevant, however, when considering the different arrangements, the types of tiles come into play. One thing to note is on the rules for Carcassonne it states that "The river cannot turn twice in the same direction because this would create a U-turn" (Wrede, 1995). The rule book displayed an example, but the example to some appeared to be a U-turn.



Wrede,1995

Now this led to two interpretations found on the Internet. The first interpretation states that no U-turn is allowed no matter how spread out the tiles are. The second interpretation states that no adjacent tiles can turn in the same direction, meaning no immediate or sharp turns are allowed. In order to calculate the total number of configurations of the river expansion pack, both interpretations must be calculated.

4 Alternating Turns

The first of the interpretations is Alternating Turns, meaning the first curved tile has a choice whether to turn right or left, but then all the other tiles must turn the opposite way of the previous turn. That implies there are two options to rotate the first curved tile, and one for the other curved tiles. There are also two straight tiles that, if rotated, do not affect the flow of the river but do affect how the game continues due to the features presented and lack of symmetry.

So for these tiles they can rotate two ways each. So, there are a total of $2 \cdot 2 \cdot 2 = 8$ different rotations possible for the six directed tiles. That means for the layouts of the Alternating Turns case, there are $\frac{10!}{2!2!} \cdot 23 = 7{,}257{,}600$ different layouts.

4.1 Conclusion of Alternating Turns

In the Alternating Turns interpretation, what needed to be considered were the number of stack orders and the rotation of the tiles. Overall, those who play with this interpretation in mind can expect to have 7,257,600 different configurations.

4.2 Quickening Analogy

Consider QUICKENING again. Let the consonants represent the six straight tiles, and the vowels represent the four curved tiles. To avoid immediate U-turns, the curved tiles need to be separated by a straight tile. Likewise, to avoid consecutive vowels, the vowels need to be separated by the consonants. There are 6 consonants, where two of them are interchangeable, otherwise known as the "N"'s. Therefore, when choosing where to place the consonants, note that two of them are interchangeable. That means, $\binom{6}{2}$represents the two Ns out of the six consonants. In order to find how many ways to place the consonants now with the interchangeable, it becomes $\binom{6}{2} \cdot \binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 360$. Then place the vowels. So, in order to keep them separate, the six consonants need to be placed first. For instance, consider the six slots as consonants:

$$— — — — — —.$$

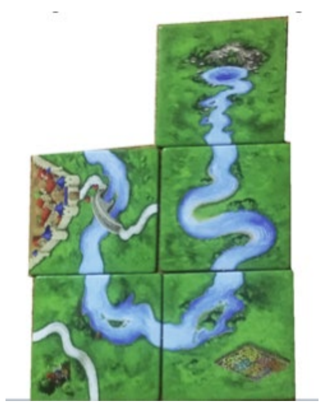
If a vowel had to be separated by a consonant in order to be placed, these are the possible locations where an arrow denotes a vowel:

$$\Uparrow—\Uparrow—\Uparrow—\Uparrow—\Uparrow—\Uparrow—\Uparrow.$$

If an arrow denotes a vowel, and there are seven arrows, that means there are seven slots for the vowels. Since there are only four vowels, that means $\binom{7}{4}$ represents the slots that the vowels can be placed. Then each individual vowel needs to be placed among the possible slots. Since there are two Is, like the two Ns, that is represented by $\binom{4}{2}$. The other vowels are distinct, therefore they are represented by $\binom{2}{1} \cdot \binom{1}{1}$. All together, with eleven those multiplied together, $\binom{7}{4} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1}$= 420. In order to determine how many different ways to arrange quickening without have consecutive vowels, multiply the number of ways to place the consonants and vowels, so $360 \cdot 420 = 151,200$ ways to arrange quickening with no consecutive vowels.

5 No U-turns Interpretation

The no U-turn interpretation means that tiles can be adjacent, however the tiles cannot turn in the same direction, meaning if a curved tile is adjacent to the other, it must rotate in the opposite direction.



In this case, it is important to note the rotations will change for each case based off of if the tiles are adjacent or not. With that in mind, there are five cases that need to be calculated.

There are five cases to consider within the interpretation:

- Case 1: No adjacent curved tiles. All four curved tiles are separated by a straight tile.
- Case 2: Two curved tiles are adjacent; the rest are separated.
- Case 3: Two pairs of adjacent curves. The pairs are separated by a straight tile.

- Case 4: Four curved tiles are adjacent, none are separated by a straight tile.
- Case 5: Three curved tiles are adjacent, the fourth is separated by a straight tile.

The reason these cases need to be considered is because for each case there is a different layout that is created. In order to calculate the total number of layouts possible for the river tile expansion pack in this interpretation, each case needs to be accounted for.

5.1 Case 1

Consider the ten tiles, excluding the starting and ending tile. How many different possibilities are there if no two curved tiles are connected? There are four curved tiles and six straight tiles. The number of ways to place the curved tiles is $\binom{7}{4}$. Determining the exact slots the 4 tiles are placed is $\binom{7}{4} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 420$. Then there are still the straight tiles to choose, so $\binom{6}{2} \cdot \binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 360$. In total, for case 1 there are $420 \cdot 360 = 151{,}200$ possible arrangements not including rotation of tiles. This is very similar to quickening. Since all the curved tiles are separated by a straight tile, they each have two rotations for the layout for a total of $2^6 = 64$ orientations for the six directed tiles. Therefore, for Case 1 there are $151200 \cdot 64 = 9{,}676{,}800$ different layouts.

5.2 Case 2

In this case, there are two adjacent curved tiles and two separate curve tiles. Consider the two adjacent curved tiles as one tile, considering they must be near one another. There are still six straight tiles, so 360 possible arrangements for the straight tiles. When placing the curved tiles, instead of placing four like in Case 1, only three need to be placed since two are adjacent, or considered "one" tile. Therefore, to find the possible slots, it is $\binom{7}{3}$. Now, to find which goes where is similar to Case 1, because there are still 3 interchangeable tiles and distinct tiles is $\binom{7}{3} \cdot \binom{3}{1} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 1{,}260$. Then $1{,}260 \cdot 360 = 453{,}600$. Since in this case there are two adjacent tiles, the first tile has a choice of how it is placed, but the second must turn in the opposite direction in order to avoid this U-turn. The other six directed tiles are not affected by this; therefore, they still have two rotations. That means for the six directed tiles, there are 2^5 =32 orientations of the directed tiles for a grand total of $1{,}260 \cdot 360 \cdot 32 = 14{,}515{,}200$ layouts for Case 2.

5.3 Case 3

For this next case, there are two pairs of adjacent tiles, meaning two curved tiles are adjacent. Then there is a straight tile that separates another two adjacent curved tiles. This case is similar to Case 2. The number of layouts is represented by $\binom{7}{2} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1}$ = 252. then, 2^4 = 16 rotations for the six directed pieces. This is because the first curved tile of each pair has a choice on how it is placed, and then the two straight directed tiles always have two options for how they are placed. That means there are $252 \cdot 360 \cdot 16 = 2{,}903{,}040$ different configurations for this case.

5.4 Case 4

Three curved tiles are adjacent, and one is separated by a straight tile. In this case, consider 3 tiles as one tile so that they must be adjacent to one another. Therefore, in order to choose the possible slots to place the curve tiles there are $\binom{7}{2}$ = 21. Which then means, $\binom{7}{2} \cdot \binom{2}{1} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1}$ = 504.

In this case, the first curved tile of the adjacent trio has an option for how it is rotated, but the other two are dependent on that first tile. Then the fourth curved tile is separated by straight tiles so it has two options. As always, the two directed straight tiles have two options of how they are oriented. This is a total of 2^4 = 16 orientations. Hence, for Case 3 there are $504 \cdot 360 \cdot 16 = 1{,}451{,}520$ different configurations of the river expansion pack.

5.5 Case 5

In this case, all four curved tiles are adjacent, so there are none separated by a straight tile. Similar to the cases before, consider the four tiles as one tile. Therefore $\binom{7}{1} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1}$ = 84. In this case, only one curved tile has a choice of how it can be placed. The other three, since they are adjacent, are dependent on the previous tile to avoid an immediate U-turn. This means there are only 2^3 different rotations for the directed tiles, and a total of $84 \cdot 360 \cdot 8 = 241,920$ layouts.

5.6 Conclusion for No U-Turn Cases

These cases appear similar to the Alternating Turns case; however, the way the directed tiles are placed had more of an effect on the layout and number of configurations of the game. There are a total of 28,788,480 configurations for this interpretation.

6 Alternating Versus No U-Turn

These two interpretations resulted in five cases to discover and calculate how many different layouts can be made with the river expansion pack. In the Alternating Turns case there were a total of 7,257,600 possible layouts. In the No Immediate U-Turn interpretation there were 28,788,480 different configurations available. It is evident that in the No Immediate U-turn interpretation there are more layouts and configurations available. Whether that means it is better to play with that interpretation verse the Alternating Turns interpretation is up to the players.

7 Matlab Code

The article *Carcassonne In the Classroom* had created a simulation for the main game of Carcassonne. In this simulation, it randomly picked tiles and placed them on a board, given no restrictions of board space. This was to simulate a large number of games, such as 500, instead of constantly playing the game. The code did not account for the river expansion pack.

In order to simulate the river expansion pack, each tile must be categorized by its features, such as river, meadow, road, and city. In order to categorize these features, each edge of the tiles is observed to categorize it. *Carcassonne In the Classroom* had categorized their tiles similarly, excluding the river, since there is no river in the main game. They started by looking at the top edge and moving clockwise around the edges of the tiles, where 1 represented road, 2 is city, 3 is meadow, and in addition for the river expansion pack, 4 is river.

A list of all the tiles are then created in a 12 by 4 matrix, where 12 represents each new river tiles, and four is the four corners or edges of the individual tiles.

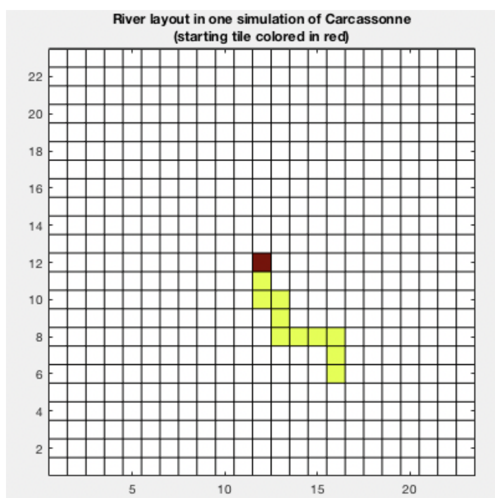
In order to make sure the order of the tiles was random each time, a random permutation had to be simulated. This made it so that every time a "player" would pick a tile from the stack, the stack was different and each tile was equally likely of being the first or the last tile picked. This excluded the starting and ending tile, because in order to play the river expansion pack, the starting tile must always be placed first and the ending tile must always be placed last.

Another thing to consider is the maximum board space needed to play this game. If all the tiles were laid out top to bottom, there are twenty-four slots, and same with side to side. However, consider the middle tile always in the center so there are 23 by 23 slots for the board, with the starting tile placed at 12, 12. Now comes the time to start placing our tiles, or discovering where to place our tiles.

Consider the starting tile is placed. The first tile will always have three meadows and one river. In the river tile expansion pack, rivers must match up with rivers. The first thing to consider is where can a tile be placed. Each river must be looked at to see where there is a "river out" meaning the river is leaving that tile. After that, then the spot for the next tile to be placed is chosen. Since the spot is chosen for the next tile, the exact tile that gets placed in that spot needs to be chosen. This is where the random permutation comes into effect. A random tile is chosen, and it needs to be discovered where the "river in" is, meaning where can the river be attached to the river in the previous tile. For some tiles, they must be rotated a few times in order to make the rivers connect. Therefore, there must be a loop that rotates the tile until the river in connects with the previous tiles' river out.

That may seem easy enough, but there are still more factors to consider, one being which interpretation is going to be used when accounting for the curved tiles. How can the code distinguish between curved tiles? The Alternating Turns interpretation was used for code. That is used to prevent discarding tiles. That means when a tile location is decided, and a tile is picked, then each edge must be observed to note if the river in and river out are on adjacent edges. If they are, then the tile is a curved tile. If it is not, then the tile can be placed, and the simulation continues. If it is a curved tile, there is more to consider.

First, in which "direction" is the tile curved? Is it a left turn or a right turn? This is why it matters which is the river in and the river out. First note where the river in is. Then, from there note where the river out is. If the river out is "left" of the river in, then it is a left turn, if the river out is "right" of the river in, then it is a right turn. This is done for each curved tile. As mentioned earlier, the interpretation used for the simulation is Alternating Turns. That means each time a curved tile is placed, it needs to be noted if it is a right turn or a left turn, this was done by a counter for right turn and left turn. If a left turn is already placed and the next curved tile is found to be a left turn, then this new tile must be rotated. This new tile has to be rotated such that river in still matches the previous tile's river out, and it cannot be the same turn as the previous curved tile. After all that, then it can finally be placed, and a configuration is outputted.



8 Future Work

In addition to those two interpretations, there is a third interpretation that had not been considered. This interpretation is similar to the Alternating Turns interpretation but would allow for more configurations of the board game. This is the no three in a row interpretation, which means that there can be two left turns but not three in a layout or two right turns but not three in a layout. This is because it would still prevent a tile not being able to be placed and allow for more configurations. In addition to a new interpretation, it would also be important to combine the river expansion Matlab code with the main game Carcassonne Matlab code created by Capaldi and Kolba (2017). This way instead of having simulations of the main game or simulations of the river expansion pack, it would be an overall simulation of the game. The new interpretation and expanding the Matlab code are two mathematical projects that have yet to be considered and present possibilities for future work.

9 Conclusion

The river expansion pack proved to have many mathematical aspects to it. The number of layouts appeared to be simple, but when the rulebook was open to many interpretations, different cases were created. Likewise, in order to see examples of these layouts at a fast pace, Matlab code proved itself helpful in order to simulate games of the river expansion pack. Carcassonne led itself to mathematics while still being a competitive and fun game.

10 Acknowledgements

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