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## Semi-direct Galois covers of the affine line

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**Abstract.** Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $G$  be a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$  where  $b$  is a positive integer and  $\ell$  is a prime distinct from  $p$ . In this paper, we study Galois covers  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group  $G$ . We find the minimal genus of a curve  $Z$  which admits a covering map of this form and we give an explicit formula for this genus in terms of  $\ell$  and  $p$ . The minimal genus occurs when  $b$  equals the order  $a$  of  $\ell$  modulo  $p$  and we also prove that the number of curves  $Z$  of this minimal genus which admit such a covering map is at most  $(p-1)/a$  when  $p$  is odd.

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## 1 Introduction

Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . In sharp contrast with the situation in characteristic 0, there exist Galois covers  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over infinity. By Abhyankar's Conjecture [2], proved by Raynaud and Harbater [9], [4], a finite group  $G$  occurs as the Galois group of such a cover  $\psi$  if and only if  $G$  is quasi- $p$ , i.e.,  $G$  is generated by  $p$ -groups. This result classifies all the finite quotients of the fundamental group  $\pi_1(\mathbb{A}_k^1)$ . It does not, however, determine the profinite group structure of  $\pi_1(\mathbb{A}_k^1)$  because this fundamental group is an infinitely generated profinite group.

There are many open questions about Galois covers  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over infinity. For example, given a finite quasi- $p$  group  $G$ , what is the smallest integer  $g$  for which there exists a cover  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over infinity with  $Z$  of genus  $g$ ? As another example, suppose  $G$  and  $H$  are two finite quasi- $p$  groups such that  $H$  is a quotient of  $G$ . Given an unramified Galois cover  $\phi$  of  $\mathbb{A}_k^1$  with group  $H$ , under what situations can one dominate  $\phi$  with an unramified Galois cover  $\psi$  of  $\mathbb{A}_k^1$  with Galois group  $G$ ? Answering these questions will give progress towards understanding how the finite quotients of  $\pi_1(\mathbb{A}_k^1)$  fit together in an inverse system. These questions are more tractable for quasi- $p$  groups that are  $p$ -groups since the maximal pro- $p$  quotient  $\pi_1^p(\mathbb{A}_k^1)$  is free (of infinite rank) [10].

In this paper, we study Galois covers  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  whose Galois group is a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ , where  $\ell$  is a prime distinct from  $p$ . Such a cover  $\psi$  must be a composition  $\psi = \phi \circ \omega$  where  $\omega : Z \rightarrow Y$  is unramified and  $\phi : Y \rightarrow \mathbb{P}_k^1$  is an Artin-Schreier cover ramified only over  $\infty$ . The cover  $\phi$  has an affine equation  $y^p - y = f(x)$  for some  $f(x) \in k[x]$  with degree  $s$  prime-to- $p$ . The  $\ell$ -torsion  $\text{Jac}(Y)[\ell]$  of the Jacobian of  $Y$  is isomorphic to  $(\mathbb{Z}/\ell\mathbb{Z})^{2g_Y}$ . When  $f(x) = x^s$ , we determine how an automorphism  $\tau$  of  $Y$  of order  $p$  acts on  $\text{Jac}(Y)[\ell]$ . This allows us to construct a Galois cover  $\psi_a : Z_a \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  which dominates  $\phi$ , such that the Galois group of  $\psi_a$  is  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$  where  $a$  is the order of  $\ell$  modulo  $p$  (Section 3). We prove that the genus of  $Z_a$  is minimal among all natural numbers that occur as the genus of a curve  $Z$  which admits a covering map  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ . We also prove that the number of curves  $Z$  of this minimal genus which admit such a covering map is at most  $(p-1)/a$  when  $p$  is odd (Section 4).

## 2 Quasi- $p$ semi-direct products

We recall which groups occur as Galois groups of covers of  $\mathbb{P}_k^1$  ramified only over  $\infty$ .

**Definition 2.1** *A finite group is a quasi  $p$ -group if it is generated by all of its Sylow  $p$ -subgroups.*

It is well-known that there are other equivalent formulations of the quasi- $p$  property, such as the next result.

**Lemma 2.2** *A finite group is a quasi  $p$ -group if and only if it has no nontrivial quotient group whose order is relatively prime to  $p$ .*

The importance of the quasi- $p$  property is that it characterizes which finite groups occur as Galois groups of unramified covers of the affine line.

**Theorem 2.3** *A finite group occurs as the Galois group of a Galois cover of the projective line  $\mathbb{P}_k^1$  ramified only over infinity if and only if it is a quasi- $p$  group.*

**Proof** This is a special case of Abhyankar's Conjecture [2] which was jointly proved by Harbater [4] and Raynaud [9].  $\square$

We now restrict our attention to groups  $G$  that are semi-direct products of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ . The semi-direct product action is determined by a homomorphism  $\iota : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^b)$ .

**Lemma 2.4** *Suppose a quasi- $p$  group  $G$  is a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$  for a positive integer  $b$ .*

1. *Then  $G$  is not a direct product.*
2. *Moreover,  $b \geq \text{ord}_p(\ell)$  where  $\text{ord}_p(\ell)$  is the order of  $\ell$  modulo  $p$ .*

**Proof** Part (1) is true since  $(\mathbb{Z}/\ell\mathbb{Z})^b$  cannot be a quotient of the quasi- $p$  group  $G$ . For part (2), the structure of a semi-direct product  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$  depends on a homomorphism  $\iota : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^b)$ . By part (1),  $\iota$  is an inclusion. Thus  $\text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^b) \simeq \text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})$  has an element of order  $p$ . Now

$$|\text{GL}_b(\mathbb{Z}/\ell\mathbb{Z})| = (\ell^b - 1)(\ell^b - \ell) \cdots (\ell^b - \ell^{b-1}).$$

Thus  $\ell^\beta \equiv 1 \pmod{p}$  for some positive integer  $\beta \leq b$  which implies  $b \geq \text{ord}_p(\ell)$ .  $\square$

**Lemma 2.5** *If  $a = \text{ord}_p(\ell)$ , then there exists a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$  which is quasi- $p$ . It is unique up to isomorphism.*

**Proof** If  $a = \text{ord}_p(\ell)$ , then there is an element of order  $p$  in  $\text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^a)$  and so there is an injective homomorphism  $\iota : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^a)$ . Thus there exists a non-abelian semi-direct product  $G$  of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ . To show that  $G$  is quasi- $p$ , suppose  $N$  is a normal subgroup of  $G$  whose index is relatively prime to  $p$ . Then  $N$  contains an element  $\tau$  of order  $p$ . By [3, 5.4, Thm. 9], since  $G$  is not a direct product and  $(\mathbb{Z}/\ell\mathbb{Z})^a$  is normal in  $G$ , the subgroup  $\langle \tau \rangle$  is not normal in  $G$ . Thus  $\langle \tau \rangle$  is a proper subgroup of  $N$ . It follows that  $\ell$  divides  $|N|$  and so  $N$  contains an element  $h$  of order  $\ell$  by Cauchy's theorem. Recall that  $\text{Aut}((\mathbb{Z}/\ell\mathbb{Z})^\beta)$  contains no element of order  $p$  for any positive integer  $\beta < a$ . Thus the group generated by the conjugates of  $h$  under  $\tau$  has order divisible by  $\ell^a$ . Thus  $N = G$  and  $G$  has no non-trivial quotient group whose order is relatively prime to  $p$ . By Lemma 2.2,  $G$  is quasi- $p$ .

The uniqueness follows from [8, Lemma 6.6].  $\square$

### 3 Explicit construction of $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ -Galois covers of $\mathbb{A}_k^1$

In this section, we give concrete examples of Galois covers  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ . To compute the genus of the covering curve  $Z$ , we will need to determine the higher ramification groups of  $\psi$ .

**Definition 3.1** *Let  $L/K$  be a Galois extension of function fields of curves with Galois group  $G$  and let  $P, P'$  be primes of  $K$  and  $L$  such that  $P'|P$ . Let  $\nu_{P'}$  and  $\mathcal{O}_{P'}$  be the corresponding valuation function and valuation ring for  $P'$ . For any integer  $i \geq -1$ , the  $i$ th ramification group of  $P'|P$  is*

$$I_i(P'|P) = \{\sigma \in G \mid \nu_{P'}(\sigma(z) - z) \geq i + 1, \forall z \in \mathcal{O}_{P'}\}.$$

**Lemma 3.2** *Suppose  $f(x) \in k[x]$  is a polynomial of degree  $s$  for a positive integer  $s$  prime to  $p$ . Let  $\phi : Y \rightarrow \mathbb{P}_k^1$  be the cover of curves corresponding to the field extension*

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)).$$

1. Then  $\phi : Y \rightarrow \mathbb{P}_k^1$  is a Galois cover with Galois group  $\mathbb{Z}/p\mathbb{Z}$  ramified only at the point  $P_\infty$  over  $\infty$ .
2. The  $i$ th ramification group at  $P_\infty$  satisfies

$$I_i = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \leq s \\ 0 & \text{if } i > s. \end{cases}$$

3. The genus  $g_Y$  of  $Y$  is equal to

$$g_Y = (p-1)(s-1)/2.$$

**Proof** For part (1), note that the extension  $k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x))$  is cyclic of degree  $p$ , with Galois group generated by the automorphism  $\tau : y \mapsto y + 1$  of order  $p$ . Let  $P$  be a finite prime of  $k(x)$  and let  $\nu_P$  be the corresponding valuation. Then  $\nu_P(f(x)) \geq 0$ , hence  $P$  is unramified by [12, Prop. III.7.8(b)]. For the infinite prime  $\infty$  with corresponding valuation  $\nu_\infty$ , we have

$$\nu_\infty(f(x) - (z^p - z)) \leq 0$$

for all  $z \in k[x]$  thus  $P_\infty$  is totally ramified by [12, Prop. III.7.8(c)].

To prove part (2), we note that furthermore

$$v_{P_\infty}(y^p - y) = v_{P_\infty}(f(x)) = v_{P_\infty}(x^s) = -sp,$$

which implies that

$$v_{P_\infty}(y) = -s.$$

Now let  $\hat{\theta}$  be the completion of the valuation ring of  $k(x)[y]/(y^p - y - f(x))$  at  $P_\infty$ , and let  $\pi_\infty$  be a generator of the unique prime in  $\hat{\theta}$ . Then write  $y = \pi_\infty^{-s}u$ , where  $u$  is a unit in  $\hat{\theta} \simeq k[[\pi_\infty]]$ . Since  $k$  is algebraically closed,  $\sqrt[s]{u} \in \hat{\theta}$ , and so  $\sqrt[s]{y} \in \hat{\theta}$ . After possibly changing  $\pi_\infty$ , we can assume without loss of generality that  $\sqrt[s]{y} = \pi_\infty^{-1}$ . Recalling that  $\tau$  acts on  $y$  by  $\tau(y) = y + 1$ , we have

$$\begin{aligned} \tau(\pi_\infty) &= \tau(1/y)^{1/s} = (\pi_\infty^s / (1 + \pi_\infty^s))^{1/s} \\ &= \pi_\infty (1 - \pi_\infty^s + \pi_\infty^{2s} - \dots)^{1/s} \\ &= \pi_\infty - (1/s)\pi_\infty^{s+1} + a_{2s+1}\pi_\infty^{2s+1} - \dots \end{aligned}$$

Thus  $v_{P_\infty}(\tau(\pi_\infty) - \pi_\infty) = s + 1$ , which completes the proof of part (2).

To find the genus  $g_Y$  of  $Y$  for part (3), we make use of the Riemann-Hurwitz formula

$$2g_Y - 2 = p(-2) + \sum_{i=0}^{\infty} (|I_i| - 1),$$

where  $I_i$  denotes the  $i$ th ramification group at  $P_\infty$ , [5, Thms. 7.27 & 11.72]). From part (2), we then obtain that  $g_Y = (p-1)(s-1)/2$ .  $\square$

Recall the following facts about the  $p$ th cyclotomic polynomial  $\Phi_p(t) := t^{p-1} + \dots + 1$ , which is the minimal polynomial over  $\mathbb{Q}$  of a primitive  $p$ th root of unity  $\zeta_p$ . Now  $\mathbb{Q}(\zeta_p)$  is a Galois extension of  $\mathbb{Q}$ , unramified over  $\ell$  since  $\ell \neq p$ , and all primes over  $\ell$  have the same residue field degree. The irreducible factors of  $\Phi_p(t)$  modulo  $\ell$  are in one-to-one correspondence with the primes of  $\mathbb{Z}[\zeta_p]$  over  $\ell$ , and each of their degrees is equal to the residue field degree of the corresponding prime over  $\ell$ . The latter equals the order  $a = \text{ord}_p(\ell)$  of  $\ell$  modulo  $p$  [3, Ch. 12.2, Exercise #20].

We shall soon explicitly construct a cover of  $\mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ . But before we do so, we start with a specific example.

**Example 3.3** Let  $p$  be an odd prime. Consider the Artin-Schreier cover  $\phi : Y_2 \rightarrow \mathbb{P}_k^1$  corresponding to the field extension  $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^2)$ . By Lemma 3.2(3), the genus of  $Y_2$  is  $g_Y = (p-1)/2$ .

Let  $\text{Jac}(Y)$  be the Jacobian of  $Y$ . The automorphism  $\tau$  of  $Y$  given by  $\tau(y) = y + 1$  defines an automorphism of  $\text{Jac}(Y)$  of order  $p$ .

Now we describe the action of  $\tau$  on the subgroup  $\text{Jac}(Y)[2]$  of 2-torsion points of  $\text{Jac}(Y)$  explicitly. Note that since  $2g_Y = (p-1)$ , then  $\text{Jac}(Y)[2]$  is isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{p-1}$  by [7, pg. 64]. Thus we can represent  $\tau$  as an element of  $\text{GL}_{p-1}(\mathbb{Z}/2\mathbb{Z})$ .

For  $0 \leq i \leq p-1$ , let  $P_i$  denote the closed point of  $Y$  at which the function  $y - i$  vanishes. For each  $i$ , the divisors  $P_i$  and  $D_i = P_i - P_\infty$  on  $Y$  can be identified with elements of  $\text{Jac}(Y)$ . Let  $O$  be the identity element of  $\text{Jac}(Y)$ , i.e., the linear equivalence class of principal divisors. Then the divisor  $2D_i$  is equivalent to  $O$  since  $\text{div}(y - i) = 2D_i$ . Moreover since  $\text{div}(x) = D_0 + D_1 + \dots + D_{p-1}$  is equivalent to 0, we have  $D_i \in \text{Jac}(Y)[2]$  with the only relation  $D_{p-1} = -(D_0 + D_1 + \dots + D_{p-2})$ . In particular,  $D_0, \dots, D_{p-2}$  form a basis of  $\text{Jac}(Y)[2]$ . With respect to this basis, the action of  $\tau$  can be represented by the  $(p-1) \times (p-1)$ -matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & 0 & -1 \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial of  $\tau$  is  $\Phi_p(t) = 1 + t + \dots + t^{p-1} \in (\mathbb{Z}/2\mathbb{Z})[t]$ , which factors into irreducible polynomials each of degree equaling the order of 2 modulo  $p$ . In particular,  $\tau$  acts irreducibly on  $\text{Jac}(Y)[2]$  if and only if 2 is a primitive root modulo  $p$ , i.e., if and only if  $p$  is an Artin prime.

For example, if  $p = 3$ , then  $\tau$  acts irreducibly on  $\text{Jac}(Y)[2]$  with minimal polynomial  $\Phi_3(t) = t^2 + t + 1$ . If  $p = 7$ , then 2 has order 3 modulo 7 and the factorization of  $\Phi_7(t)$  into irreducible polynomials is  $\Phi_7(t) \equiv (x^3 + x^2 + 1)(x^3 + x + 1)$  modulo 2. Thus the action of  $\tau$  on  $\text{Jac}(Y)[2]$  can be represented by the  $6 \times 6$ -matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where  $A_1$  and  $A_2$  are the irreducible 3-dimensional companion matrices of  $x^3 + x^2 + 1$  and  $x^3 + x + 1$  respectively.

For the rest of the paper, let  $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$  be the Artin-Schreier cover corresponding to the field extension

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s).$$

We show that  $\phi_s$  can be dominated by a Galois cover of  $\mathbb{P}_k^1$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$  for  $a$  equal to the order of  $\ell$  modulo  $p$ .

**Proposition 3.4** *Let  $s$  and  $\ell$  be primes distinct from  $p$ . Let  $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$  be the Artin-Schreier cover with affine equation  $y^p - y = x^s$ . Let  $a = \text{ord}_p(\ell)$  be the order of  $\ell$  modulo  $p$ . Then there exists an unramified Galois cover  $\omega : Z_a \rightarrow Y_s$  with Galois group  $(\mathbb{Z}/\ell\mathbb{Z})^a$  such that  $\psi_a = \phi_s \circ \omega : Z_a \rightarrow \mathbb{P}_k^1$  is a Galois cover of  $\mathbb{P}_k^1$  ramified only over  $\infty$  whose Galois group is a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ .*

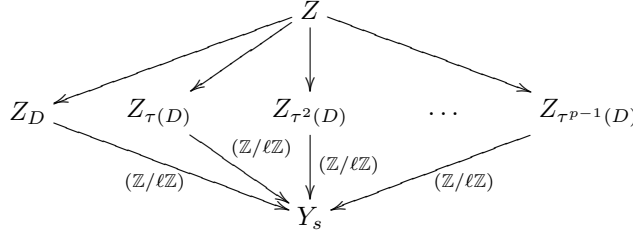
**Proof** By Lemma 3.2(1),  $\phi_s : Y_s \rightarrow \mathbb{P}_k^1$  is a Galois cover with Galois group  $\mathbb{Z}/p\mathbb{Z}$  ramified only at the point  $P_\infty$  over  $\infty$ . The genus  $g_s$  of  $Y_s$  is  $(p-1)(s-1)/2$ . Consider

two commuting automorphisms of  $Y_s$  defined by

$$\tau : \begin{cases} x \mapsto x, \\ y \mapsto y + 1, \end{cases} \quad \sigma : \begin{cases} x \mapsto \zeta_s x, \text{ where } \zeta_s \text{ is a primitive } s\text{th root of unity,} \\ y \mapsto y. \end{cases}$$

Let  $\text{Jac}(Y_s)$  be the Jacobian of  $Y_s$ . Then  $\tau$  and  $\sigma$  define commuting automorphisms of  $\text{Jac}(Y_s)$  of orders  $p$  and  $s$  respectively. Therefore,  $\text{End}(\text{Jac}(Y_s))$  contains a ring isomorphic to  $\mathbb{Z}[\zeta_p, \zeta_s] \cong \mathbb{Z}[\zeta_{ps}]$ , which is a  $\mathbb{Z}$ -module of rank  $\phi(ps) = (p-1)(s-1) = 2g_s$ . Then  $\mathbb{Q}(\zeta_{ps})$  is contained in  $\text{End}(\text{Jac}(Y_s)) \otimes \mathbb{Q}$ . In other words,  $\text{Jac}(Y_s)$  has complex multiplication by  $\mathbb{Q}(\zeta_{ps})$ .

For a prime  $\ell$  distinct from  $p$ , the automorphism  $\tau$  induces an action on the subgroup  $\text{Jac}(Y_s)[\ell]$  of  $\ell$ -torsion points of  $\text{Jac}(Y_s)$ . Recall that there is a bijection between  $\ell$ -torsion points  $D$  of  $\text{Jac}(Y_s)$  and unramified  $(\mathbb{Z}/\ell\mathbb{Z})$ -Galois covers  $\omega_D : Z_D \rightarrow Y_s$  [6, Prop. 4.11]. Also  $D$  has order  $\ell$  if and only if  $Z_D$  is connected. This induces a bijection between orbits of  $\tau$  on the set of unramified  $(\mathbb{Z}/\ell\mathbb{Z})$ -Galois covers  $\omega_D : Z_D \rightarrow Y_s$  and on the set of  $\ell$ -torsion points of  $\text{Jac}(Y_s)$ . For a point  $D$  of order  $\ell$  of  $\text{Jac}(Y_s)$ , consider the compositum  $\omega : Z \rightarrow Y_s$  of all of the conjugates  $\omega_{\tau^j(D)} : Z_{\tau^j(D)} \rightarrow Y_s$  for  $0 \leq j \leq p-1$ :



Then  $Z$  is invariant under  $\tau$  and so  $\phi_s \circ \omega : Z \rightarrow \mathbb{P}_k^1$  is Galois. Moreover,  $\phi_s \circ \omega$  is the Galois closure of  $\phi_s \circ \omega_D : Z_D \rightarrow \mathbb{P}_k^1$ .

Suppose there is a non-trivial one-dimensional  $\tau$ -invariant subspace of  $\text{Jac}(Y_s)[\ell]$  with eigenvalue 1; i.e.  $\tau$  acts trivially on this subgroup of order  $\ell$ . This yields a cover  $\psi_s \circ \omega_1 : Z_1 \rightarrow Y_s \rightarrow \mathbb{P}_k^1$ . Since the action of  $\tau$  is trivial,  $\psi_s \circ \omega_1$  is Galois, ramified only over  $\infty$ , with abelian Galois group  $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ . This contradicts Lemma 2.4.

Since  $\tau$  has order  $p$ , the minimal polynomial  $m_\tau(t)$  of  $\tau$  divides  $t^p - 1 = (t-1)(t^{p-1} + \dots + 1)$  in  $(\mathbb{Z}/\ell\mathbb{Z})[t]$ . From the preceding paragraph, there is no non-trivial one-dimensional  $\tau$ -invariant subspace of  $\text{Jac}(Y_s)[\ell]$  with eigenvalue 1. This implies that  $m_\tau(t)$  divides the  $p$ th cyclotomic polynomial  $\Phi_p(t) = t^{p-1} + \dots + 1$  in  $(\mathbb{Z}/\ell\mathbb{Z})[t]$ . The irreducible factors of  $\Phi_p(t)$  in  $(\mathbb{Z}/\ell\mathbb{Z})[t]$  all have degree  $a$ . Thus the degree of  $m_\tau(t)$  equals  $a$ .

Since  $2g_s = (p-1)(s-1)$ , we have  $\text{Jac}(Y_s)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{(p-1)(s-1)}$ , so we can represent  $\tau$  as an element of  $\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$ . We can choose a basis of  $\text{Jac}(Y_s)[\ell]$  such that  $\tau$  is represented as an element of  $\text{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$  in block form. The first irreducible subrepresentation of  $\tau$  has dimension  $a$ . Moreover, since  $\mathbb{Q}(\zeta_{ps})$  is a Galois extension of  $\mathbb{Q}$ , the block form of  $\tau$  consists entirely of irreducible blocks of the same size. In particular, the number of irreducible blocks is  $(p-1)(s-1)/a$ . In other words,  $\tau$  can be represented by an element of  $\text{GL}_{(s-1)(p-1)}(\mathbb{Z}/\ell\mathbb{Z})$  of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{(p-1)(s-1)/a} \end{pmatrix},$$



where  $A_i$  is an  $a \times a$  matrix representing an  $a$ -dimensional irreducible subrepresentation of  $\tau$  on  $\text{Jac}(Y_s)[\ell]$ .

Using the bijection between orbits of  $\text{Jac}(Y_s)[\ell]$  and orbits of  $(\mathbb{Z}/\ell\mathbb{Z})$ -covers of  $Y_s$  under  $\tau$  and the above observation for the action of  $\tau$  on  $\text{Jac}(Y_s)[\ell]$ , there exists an unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover  $\omega : Z_a \rightarrow Y_s$  such that  $\psi_a = \phi_s \circ \omega : Z_a \rightarrow \mathbb{P}_k^1$  is a Galois cover of  $\mathbb{P}_k^1$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ . Also  $\psi_a$  is ramified only over infinity since  $\phi_s$  is ramified only over  $\infty$  and since  $\omega$  is unramified.  $\square$

#### 4 Minimal genus of $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ -Galois covers of $\mathbb{A}_k^1$

In this section, we find the minimal genus of a curve  $Z$  that admits a covering map  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$ , with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ . The minimal genus depends only on  $\ell$  and  $p$ . We consider the cases  $p$  odd and  $p = 2$  separately. We also prove that the number of curves  $Z$  of this minimal genus which admit such a covering map is at most  $(p-1)/a$  when  $p$  is odd and at most  $\ell + 1$  when  $p = 2$ . The following lemma will be useful.

**Lemma 4.1** *Let  $G$  be a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$  where  $\ell$  is a prime distinct from  $p$ . If  $\psi : Z \rightarrow \mathbb{P}_k^1$  is a Galois cover ramified only over  $\infty$  with Galois group  $G$ , then the subcover  $\omega : Z \rightarrow Y$  with Galois group  $(\mathbb{Z}/\ell\mathbb{Z})^b$  is unramified.*

**Proof** The quotient of  $G$  by the normal subgroup  $N = (\mathbb{Z}/\ell\mathbb{Z})^b$  is  $\mathbb{Z}/p\mathbb{Z}$ . Thus the cover  $\psi$  is a composition  $\psi = \phi \circ \omega$  where  $\phi : Y \rightarrow \mathbb{P}_k^1$  has Galois group  $\mathbb{Z}/p\mathbb{Z}$  and is totally ramified at the unique point  $P_\infty$  over  $\infty$  and where  $\omega : Z \rightarrow Y$  has Galois group  $N$  and is branched only over  $P_\infty$ . Then  $\omega$  is a prime-to- $p$  abelian cover of  $Y$ . Let  $g$  be the genus of  $Y$ . Then by [1, XIII, Cor. 2.12], the prime-to- $p$  fundamental group of  $Y - \{P_\infty\}$  is isomorphic to the prime-to- $p$  quotient  $\Gamma$  of the free group on generators  $\{a_1, b_1, \dots, a_g, b_g, c\}$  subject to the relation  $\prod_{i=1}^g [a_i, b_i] = c^{-1}$ . The cover  $\omega$  corresponds to a surjection of  $\Gamma$  onto  $N$  where  $c$  maps to the canonical generator of inertia  $\gamma$  of a point of  $Z$  over  $P_\infty$ . Thus  $N$  is generated by elements  $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma\}$  subject to the relation  $\prod_{i=1}^g [\alpha_i, \beta_i] = \gamma^{-1}$ . Then  $\gamma = 1$  since  $N$  is abelian and so  $\omega$  is unramified.  $\square$

**Theorem 4.2** *Let  $p$  be an odd prime. Let  $\ell$  be a prime distinct from  $p$  and let  $a$  be the order of  $\ell$  modulo  $p$ . Then:*

1. *There exists a Galois cover  $\psi_a : Z_a \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  whose Galois group is a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$  such that  $g_{Z_a} = 1 + \ell^a(p-3)/2$ .*
2. *The integer  $g_{Z_a}$  is the minimal genus of a curve  $Z$  which admits a covering map  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$  for any positive integer  $b$ .*
3. *There are at most  $(p-1)/a$  isomorphism classes of curves  $Z$  which admit a Galois covering map as in part (1) with minimal genus  $g_{Z_a}$ .*

**Proof** By the construction in Proposition 3.4, there exists a Galois cover  $\psi_a : Z_a \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  whose Galois group is a semi-direct product of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ . We compute the genus of the curve  $Z_a$ . Recall that  $\psi_a$  is a composition  $\psi = \phi_2 \circ \omega$  where  $\omega : Z \rightarrow Y_2$  is an unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover and  $\phi_2 : Y_2 \rightarrow \mathbb{P}_k^1$  has Artin-Schreier equation  $y^p - y = x^2$ . Then  $Y_2$  has genus  $g_{Y_2} = (p-1)/2$  by Lemma 3.2(3). By the Riemann-Hurwitz formula,  $2g_{Z_a} - 2 = \ell^a(2g_{Y_2} - 2) = \ell^a(p-3)$ , i.e.,  $g_{Z_a} = 1 + \ell^a(p-3)/2$ . This completes part (1).

For part (2), suppose  $\psi : Z \rightarrow \mathbb{P}_k^1$  is a Galois cover ramified only over  $\infty$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ . If  $g$  is the genus of  $Z$ , we will show that  $g \geq g_{Z_a}$ . As described in the proof of Lemma 4.1, the cover  $\psi$  is a composition  $\psi = \phi \circ \omega$  where  $\phi : Y \rightarrow \mathbb{P}_k^1$  has Galois group  $\mathbb{Z}/p\mathbb{Z}$  and is ramified only over  $\infty$  and where  $\omega$  is unramified with group  $(\mathbb{Z}/\ell\mathbb{Z})^b$ . By the Riemann-Hurwitz formula,  $2g - 2 = \ell^b(2g_Y - 2)$ .

By Artin-Schreier theory,  $\phi$  is given by an equation  $y^p - y = f(x)$  where  $f \in k[x]$  has degree  $s$  for some integer  $s$  relatively prime to  $p$ . Since the genus  $g_Y$  of  $Y$  is  $(p-1)(s-1)/2$  by Lemma 3.2 (3), we should make  $s$  as small as possible. The value  $s = 1$  is impossible since then  $Y$  is a projective line and there do not exist Galois covers of the projective line ramified only over one point with Galois group  $\mathbb{Z}/\ell\mathbb{Z}$ . Thus  $s = 2$  yields the smallest possible value for  $g_Y$ , namely  $(p-1)/2$ . Recall that  $b \geq a$  by Lemma 2.4. Thus  $g \geq 1 + \ell^a(p-3)/2 = g_{Z_a}$ .

For part (3), suppose  $\psi : Z \rightarrow \mathbb{P}_k^1$  is a Galois cover ramified only over  $\infty$  with Galois group of the form  $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$  and the genus of  $Z$  satisfies  $g_Z = 1 + \ell^a(p-3)/2$ . As in part (2),  $\psi$  factors as  $\phi \circ \omega$  where  $\omega : Z \rightarrow Y$  is an unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover, where  $\phi : Y \rightarrow \mathbb{P}_k^1$  is an Artin-Schreier cover ramified only over  $\infty$ , and where  $Y$  has genus  $(p-1)/2$ . By Lemma 3.2(3),  $Y$  has an affine equation  $y^p - y = a_2x^2 + a_1x + a_0$  for some  $a_0, a_1 \in k$  and  $a_2 \in k^*$ . Since  $p$  is odd and  $k$  is algebraically closed, it is possible to complete the square and write  $a_2x^2 + a_1x + a_0 = x_1^2 + \epsilon$  where  $x_1 = \sqrt{a_2}x + a_1/2\sqrt{a_2}$ . After modifying by an automorphism of the projective line, specifically by the affine linear transformation  $x \mapsto x_1$ , the equation for  $Y$  can be rewritten as  $y^p - y = x_1^2 + \epsilon$ . Since  $k$  is algebraically closed, there exists  $\delta \in k$  such that  $\delta^p - \delta = \epsilon$ . Let  $y_1 = y - \delta$ . After the change of variables  $y \mapsto y_1$ , the curve  $Y$  is isomorphic to the curve  $Y_2$  with affine equation  $y_1^p - y_1 = x_1^2$ . Thus there is a unique possibility for the isomorphism class of the curve  $Y$ .

From the proof of Proposition 3.4, there is a bijection between  $\tau$ -invariant connected unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois covers of  $Y_2$  and orbits of  $\tau$  on points  $D$  of order  $\ell$  on  $\text{Jac}(Y_2)$ . The action of  $\tau$  on  $\text{Jac}(Y_2)[\ell]$  decomposes into  $(p-1)/a$  irreducible subrepresentations. Each of these is distinct, because the irreducible factors of  $\Phi_p(t) \in (\mathbb{Z}/\ell\mathbb{Z})[t]$  are distinct. Thus there are  $(p-1)/a$  choices for a  $\tau$ -invariant unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois cover of  $Y_2$ . Thus there are at most  $(p-1)/a$  isomorphism classes of curves  $Z$  which admit a Galois covering map as in part (1) with minimal genus  $g_{Z_a}$ .  $\square$

We note that the set of curves which are unramified  $(\mathbb{Z}/\ell\mathbb{Z})^a$ -Galois covers of  $Y_2$  may contain fewer than  $(p-1)/a$  isomorphism classes of curves.

**Theorem 4.3** *Let  $p = 2$  and let  $\ell$  be an odd prime. Then:*

1. *There exists a Galois cover  $\psi : Z \rightarrow \mathbb{P}_k^1$  ramified only over  $\infty$  with Galois group of the form  $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ .*
2. *The minimal genus of a curve  $Z$  which admits a covering map as in part (1) is  $g_Z = 1$ .*
3. *There are at most  $\ell + 1$  isomorphism classes of curves  $Z$  which admit a Galois covering map as in part (1) with minimal genus  $g_Z = 1$ .*

**Proof** Note that the order of  $\ell$  modulo 2 is  $a = 1$ . For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form  $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$  which is quasi-2. The result is then immediate from Theorem 2.3.

Suppose  $\psi : Z \rightarrow \mathbb{P}_k^1$  is a Galois cover ramified only over  $\infty$  with Galois group as in part (1). As before,  $\psi$  factors as a composition  $\phi \circ \omega$  where  $\omega : Z \rightarrow Y$  has Galois group  $\mathbb{Z}/\ell\mathbb{Z}$  and  $\phi : Y \rightarrow \mathbb{P}_k^1$  is an Artin-Schreier extension with affine equation  $y^2 - y = f(x)$  for some  $f(x) \in k[x]$  of odd degree  $s$ . By Lemma 4.1,  $\omega$  is unramified.

The minimal genus for  $Z$  will thus occur when  $s$  is as small as possible. As before,  $s = 1$  is impossible, and so  $s = 3$  is the smallest choice. In this case, by Lemma 3.2(3),  $g_Y = 1$ , i.e.,  $Y$  is an elliptic curve. By the Riemann-Hurwitz formula, the minimal genus for  $Z$  is  $g_Z = 1 + \ell(g_Y - 1) = 1$ , which completes part (2).

For part (3), since  $k$  is algebraically closed, we can complete the cube of  $f(x)$  and make the corresponding change of variables, which is a scaling and translation of  $x$ . So we can assume that  $Y$  has affine equation  $y^2 - y = x^3 + a_1x + a_0$  for some  $a_0, a_1 \in k$ . Then it follows from [11, Appendix A, Prop. 1.1c] that the  $j$ -invariant of  $Y$  is  $j(Y) = 0$  and that the discriminant is  $\Delta(Y) = (-1)^4 = 1$ . Since  $k$  is algebraically closed, by [11, Appendix A, Prop. 1.2b], all elliptic curves  $Y$  with  $j(Y) = 0$  are isomorphic over  $k$ . Thus there is a unique choice for  $Y$  up to isomorphism. Without loss of generality, we may assume that  $Y = Y_3$  has affine equation  $y^2 - y = x^3$ .

From the proof of Proposition 3.4, the action of  $\tau$  on  $\text{Jac}(Y_3)[\ell]$  decomposes into the direct sum of two 1-dimensional subrepresentations. In other words, the action of  $\tau$  is diagonal with both eigenvalues equal to  $-1$ . The number of non-trivial  $\tau$ -invariant subgroups of  $\text{Jac}(Y_3)[\ell]$  is the number of subgroups of order  $\ell$  in  $(\mathbb{Z}/\ell\mathbb{Z})^2$ , which is  $\ell + 1$ . As in Theorem 4.2, this implies that there are at most  $\ell + 1$  isomorphism classes of curves  $Z$  which admit a Galois covering map as in part (1) with minimal genus  $g_Z = 1$ .  $\square$

We note that the set of curves which are unramified  $\mathbb{Z}/\ell\mathbb{Z}$ -Galois covers of  $Y_3$  may contain fewer than  $\ell + 1$  isomorphism classes of curves.

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